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## THE PROBLEM 0F THE ANGLEBISECTORS

A DISSERTATION
SUbMitted to the faculty of the ogden graduate school of SCIENCE OF THE UNIVERSITY OF CHICAGO IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
(department of mathematics)


BY
RICHARD PHILIP BAKER

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## THE HISTORY OF THE PROBLEM

The problem of constructing a triangle when the lengths of the bisectors of the angles are given has been an outstanding problem among geometers probably from the time of Pascal ${ }^{\text {r }}$ and certainly from the time of Euler. ${ }^{2}$

Brocard ${ }^{3}$ has summed up the literature, dealing almost entirely with special cases, of which the most extensive treatment appears to be the solution of the problem when one angle is a right angle due to Marcus Baker. ${ }^{3}$ This problem is of the sixth order. Among many special treatments that appear in the smaller journals the fundamental problem of determining the character of the aIgebraic irrationality involved is not mentioned. As a result apparently conflicting statements occur as to the order of the equation concerned, this depending on accidental choice of the parameter field.

The only paper dealing with the general case where the internal bisector formulas are used is P. Barbarin's. ${ }^{\text {I }}$ The case where the external formulas are to be used and the case where two of the assigned bisectors refer to the same vertex is not treated in general in any paper known to the writer.

Barbarin proved that the general internal problem could be solved by the solution of an algebraic equation of order not greater than twelve. The irreducibility and group of the equation are not discussed, and as the equation itself is not set out explicitly further reduction of the order of the problem is not precluded.

The method of attack used by Barbarin is to solve first the problem when an angle and two bisectors are given, and to use the result as a basis for attacking the general problem. The necessary sacrifice of symmetry prevents any explicit comparison with the solution given in this paper except at the cost of labor disproportionate to the result.

The problem must have an extensive domestic history in the schools: Barbarin charges that E. Catalan was among those who have proposed it as an elementary exercise, and from a Russian scholar the writer learns that it has been there extensively used in the schools as a standard set-back for ambitious young geometers.

[^0]
## METHOD AND RESULTS

The eliminations necessary in the two most difficult cases are accomplished by a combination of the method of symmetric functions and a birational transformation. A complete theory of the rôle of birational transformations is not at hand. In these cases one of the two parameters enters the resultant in the first order. The birational transformation reduces a rather complicated equation to one of the first order in the quantity to be eliminated. A mere substitution then accomplishes the elimination. In a more general case it might always be possible to reduce by such a transformation to the order in which the parameter entered (choosing the lowest where choice is offered) and materially expedite the process. Whether this is so or not must be left an open question. In this case such a surmise led to a tentative search with successful results.

The determination of the discriminant at least as to its algebraic factors was conducted in the field of the roots (sides of the triangles) by a geometric process which gave two factors afterward transformed by the transformations of the climination to the field of the parameters. It then became possible to find all the factors and check the result.

The separation of the roots was effected by following graphically and algebraically the plane of the sides as divided by discriminantal lines through the transformations to the multiply covered plane of the parameters.

In the internal case the algebra combines with the case of three internal bisectors (which has a unique real solution for real data) the cases of one internal and two external bisectors into an equation of order ten with the general group. No further irrationality is necessarily involved, though a cubic proved a convenience. The solution is explicitly given by two methods, one with and one without this cubic. For practical approximation a tentative method not involving an elimination proved superior.

In the external case the solutions are not always real for real data. The equation is of order six and has the general group.

In the mixed case there are three permutations each involving an irreducible equation of the tenth order with the general group. The separation of the roots is accomplished by a simple graph and approximate solution by trial with the table of logarithmic sines.

Special cases of equal bisectors, isosceles and right triangles are discussed. In each case the group is determined.

## I. TIIE INTERNAL PROBLEM

To construct all triangles whose internal angle-bisectors are equal to three given lines.
In the absence of a geometrical criterion as to the possibility of constructing an assigned figure with given apparatus (whether ruler, ruler and sect carrier, ruler and compass, threebar link, etc.) resort must be had to algebra.

The formulas giving the internal angle-bisectors in terms of the sides are:

$$
\begin{align*}
& K^{2}=\frac{(a+b+c)(-a+b+c) b c}{(b+c)^{2}} \\
& L^{2}=\frac{(a+b+c)(a-b+c) c a}{(c+a)^{2}}  \tag{I}\\
& M^{2}=\frac{(a+b+c)(a+b-c) a b}{(a+b)^{2}}
\end{align*}
$$

It is to be noticed at the outset that a change in sign of $a$ leaves $K$ unchanged but changes $L$ and $M$ into the corresponding external bisectors which are given by:

$$
\begin{align*}
\bar{K}^{2} & =\frac{(a-b+c)(a+b-c) b c}{(b-c)^{2}} \\
\bar{L}^{2} & =\frac{(a+b-c)(-a+b+c) c a}{(c-a)^{2}}  \tag{2}\\
\bar{M}^{2} & =\frac{(-a+b+c)(a-b+c) a b}{(a-b)^{2}}
\end{align*}
$$

We expect then an algebraic solution of the problem to involve as necessarily introduced extraneities the cases where the assigned quantities appear as one internal and two external bisectors, but not those where one external is accompanied by two internal.

## 2. THE ORDER OF THE PROBLEM

Bezout's rule as to the order of eliminants gives 64 as the number of sets of values of $a, b$, $c$, satisfying the three equations. This includes as improper solutions any sets of values which satisfy them independently of the values of $K, L, M$. That the number of such improper solutions should properly be given as 54 is to be established.

Geometrically phrased the surfaces denoted by the equations ( 1 ) intersect in general in Io distinct points which vary with $K, L, M$, and have at certain other points contact of various orders. To obtain a proper enumeration we consider slight distortions of the original surfaces whose intersections will be in fact 64 distinct points. Since there are identical relations between the planes represented by the separate factors such a distortion must destroy these, or give independence to these planes.

Algebraically phrased we write for $a, b, c$ and the other plane factors quantities differing
arbitrarily but slightly from the critical values and determine the number of intersections in a region near the critical points.

Put

$$
K^{2}=\frac{1}{k}, \quad L^{2}=\frac{I}{l}, \quad M^{2}=\frac{1}{m}
$$

and

$$
\tau=\frac{\mathrm{I}}{(a+b+c) a b c} \quad \text { and take } a+b+c=\mathbf{1}
$$

Then

$$
k=\frac{\tau a(\mathrm{I}-a)^{2}}{\mathrm{I}-2 a}, \quad l=\frac{\tau b(\mathrm{I}-b)^{2}}{\mathrm{I}-2 b}, \quad m=\frac{\tau c(\mathrm{I}-c)^{2}}{\mathrm{I}-2 c}
$$

and

$$
a+b+c=\mathbf{1}
$$

The elimination of $c$ gives three quartics so that the 64 points are still represented.
A sufficient independence is attained if we treat the quantities $\tau, a, b, c$ as independently variable in small limits near the values determined by the equations.

The point $a=1, b=0, c=0, \tau=\infty$ is critical. Write $\tau=\frac{1}{l}, a=\mathrm{I}+a, b=\beta, c=\gamma$ and consider the region near $t=0, a=0, \beta=0, \gamma=0$.

The types of proper approximations are

$$
k t=a^{2} ; l t=\beta ; m t=\gamma .
$$

Hence for assigned $k, l, m$, there are two intersections in the region and this point is to be counted a double point for every $k, l, m$ since the values of $a, b, c$ are in the limit independent of $k, l, m$.

So also
and

$$
a=0, b=\mathrm{I}, c=0, \tau=\infty
$$

$$
a=\mathrm{o}, b=\mathrm{o}, c=\mathrm{r}, \tau=\infty .
$$

The class gives then six intersections.
The point $a=\frac{1}{2}, b=\infty, c=-\infty, \tau=0$, approached by $b=p+\frac{1}{2}, c=-p$, is critical, and writing $a=\frac{1}{2}+a, b=\frac{1}{\beta}, c=\frac{1}{\gamma}$, the proper approximations are of the types $k a=\tau, l \beta^{2}=\tau$, $m \gamma^{2}=\tau$ with four solutions and six members of the class, in all 24 intersections.

The point $a=\infty, b=\infty, c=-\infty, \tau=0$, approached by $a=b=p, c=\mathrm{I}-2 p$, gives in the same way proper approximations of the types

$$
k \alpha^{2}=\tau, l \beta^{2}=\tau, m \gamma^{2}=\tau
$$

with 8 intersections in the region and three members of the class, in all 24 intersections.
The total number of points to be counted for the contacts at these critical points is then 54 .
No other combinations of values render $k, l, m$ indeterminate and we conclude that the order of the problem is ten.

## 3. THE ELIMINATION

It is convenient to substitute for the original problem that of constructing a triangle whose internal bisectors are in a given ratio, the further construction being an affair of ruler and compass and unique save as to cyclic order.

This reduction enables us to take any relation between the sides; in general we take

$$
a+b+c=1
$$

We may then proceed to eliminate $c$ by means of this and $b$ by Bezout's or other methods, arriving at an equation for $a$.

The consideration of such an equation (which is obtained in § I5) yields, however, no view of the relation between sets of angle-bisectors and sets of sides; to obtain which the symmetric function transformation of order three is introduced.

$$
\begin{align*}
a+b+c & =x=\mathrm{I} \\
a b+b c+c a & =y \\
a b c \quad & =z
\end{align*}
$$

The determination of any quantity of which $y z$ are rational functions gives after solution of a cubic equation a unique triangle; that is as to ratios of the sides. It turns out that this cubic though convenient is not algebraically necessary, the character of the irrationality involved not being essentially altered by its introduction.

Using the symmetric function method we write

$$
\begin{array}{ll}
K^{2}+L^{2}+M^{2} & =p \\
K^{2} L^{2}+L^{2} M^{2}+M^{2} K^{2}=q  \tag{4}\\
K^{2} L^{2} M^{2} & =r
\end{array}
$$

and expressing $p q r$ in terms of $x y z$, obtain

$$
\begin{align*}
& p=\frac{x}{(x y-z)^{2}}\left[x y^{3}-2 y^{2} z+3 x^{2} y z-5 x z^{2}-x^{4} z\right], \\
& q=\frac{x^{2} z}{(x y-z)^{2}}\left[4 x y^{2}-8 y z+x^{2} z-x^{3} y\right],  \tag{5}\\
& r=\frac{x^{3} z^{2}}{(x y-z)^{2}}\left[4 x y-8 z-x^{3}\right] .
\end{align*}
$$

It is convenient to write

$$
\begin{equation*}
k=\frac{\mathrm{I}}{K^{2}}, \quad l=\frac{\mathrm{I}}{L^{2}}, \quad m=\frac{\mathrm{I}}{M^{2}} . \tag{6}
\end{equation*}
$$

We now write the symmetric functions of the angle-bisectors of order zero:

$$
\begin{align*}
& \alpha=\frac{q^{2}}{p r}=\frac{(k+l+m)^{2}}{k l+l m+m k},  \tag{7}\\
& \beta=\frac{q^{3}}{r^{2}}=\frac{(k+l+m)^{3}}{k l m},
\end{align*}
$$

and use the quantities $\alpha, \beta$ as fundamental parameters.
Choosing the scale by placing $x=\mathrm{I}$

$$
\begin{align*}
& a=\frac{\left(4 y^{2}-8 y z+z-y\right)^{2}}{\left(y^{3}-2 y^{2} z+3 y z-5 z^{2}-z\right)(4 y-8 z-1)},  \tag{8}\\
& \beta=\frac{\left(4 y^{2}-8 y z+z-y\right)^{3}}{(4 y-8 z-1)^{2}(y-z)^{2} z} .
\end{align*}
$$

The elimination can now be accomplished by means of a birational transformation:

$$
\begin{align*}
& \phi=\frac{4 z(y-2 z)}{(y-z)(4 y-8 z-1)},  \tag{9}\\
& \tau=\frac{z}{(y-z)(4 y-8 z-1)},
\end{align*}
$$

applying which we have

$$
\begin{align*}
& \beta=\frac{(1+\phi)^{3}}{\tau},  \tag{10}\\
& \alpha=\frac{4 \phi(\phi+1)^{2}(\phi-\tau)}{16 \tau^{4}+\tau^{3}(-40 \phi+16)+\tau^{2}\left(33 \phi^{2}-28 \phi\right)+\tau\left(-10 \phi^{3}+10 \phi^{2}\right)+\phi^{2}(\phi+1)^{2}} .
\end{align*}
$$

Substituting for $\tau$ and writing $\phi+\mathrm{I}=\sigma$

$$
\begin{align*}
& 16 a \sigma^{\mathrm{T0}}-40 a \beta \sigma^{8}+56 a \beta \sigma^{7}+33^{\alpha} \beta^{2} \sigma^{6}-94 a \beta^{2} \sigma^{5}  \tag{II}\\
& +\sigma^{4}\left(6{ }_{1} \alpha \beta^{2}-10 \alpha \beta^{3}+4 \beta^{3}\right)+\sigma^{3}\left(40 a \beta^{3}-4 \beta^{3}\right) \\
& +\sigma^{2}\left(-50 \alpha \beta^{3}+a \beta^{4}-4 \beta^{4}\right)+\sigma\left(20 \alpha \beta^{3}-2 a \beta^{4}+8 \beta^{4}\right) \\
& +\left(\alpha \beta^{4}-4 \beta^{4}\right)=0 .
\end{align*}
$$

The factors $\sigma^{2}(\sigma-1)$ are removed in the course of the work.
This equation will be denoted by

$$
\text { . } F(\sigma, a, \beta)=0
$$

and referred to as the symmetrical internal equation.
We note that $\sigma$ is a symmetric function of $a, b, c$ and that if $\sigma$ be given, $\phi, \tau$ are uniquely determined, $y, z$ being given by

$$
\begin{align*}
& y=\frac{-\phi(\phi-\tau+1)}{4^{\tau(\phi-\tau-1)}}  \tag{r}\\
& z=\frac{-\phi(\phi-\tau)}{4^{\tau}(\phi-\tau-1)}
\end{align*}
$$

which is unique except for the singular points and lines of the transformation:
The lines $\left.\begin{array}{rl}y-z & =0 \\ 4 y-8 z-1 & =0\end{array}\right\}$ corresponding to the pqint $\phi=\infty \quad \tau=\infty$
the line $\quad y-2 z=0 \quad$ corresponding to the point $\phi=0 \quad \tau=-1$
the line $\quad z=0$ corresponding to the point $\phi=0 \quad \tau=0$ and inversely

| the line | $\phi$ | $=0$ |  | corresponding to the point $y=0 \quad z=0$ |
| :--- | ---: | :--- | ---: | :--- |
| the line | $\tau=0$ |  | corresponding to the point $y=\infty \quad z=\infty$ |  |
| the line | $\phi-\tau-\mathrm{I}$ | $=0$ | corresponding to the point $y=2 z=\infty$. |  |

If instead of choosing $x=1$ as defining the scale we take

$$
\begin{aligned}
& 4 x y-8 z-x^{3}=1 \text { and write } \\
& x y-2 z=\kappa \quad \text { whence } x^{3}=4 \kappa-1
\end{aligned}
$$

we have by elimination of $x$ and $y$

$$
\begin{align*}
& a=-\frac{[z(4 \kappa+1)+\kappa]^{2}}{z^{2}(8 \kappa-1)+z\left(5^{\kappa}-1\right)+\kappa^{3}},  \tag{12}\\
& \beta=\frac{[z(4 \kappa+1)+\kappa]^{3}}{(4 \kappa-1)(z+\kappa)^{2} z} .
\end{align*}
$$

A simpler birational transformation is now available

$$
\begin{equation*}
\phi=\frac{4^{\kappa z}}{\kappa+z}, \tau=\frac{\left(4^{\kappa-1) z}\right.}{\kappa+z}, \tag{3}
\end{equation*}
$$

with reversions

$$
\begin{equation*}
\kappa=\frac{\phi}{4(\phi-\tau)}, \quad z=\frac{\phi}{4(\mathrm{r}-\phi+\tau)} . \tag{14}
\end{equation*}
$$

The expressions for $a \beta$ and the eliminant are the same as before, the same extraneities occurring.

This transformation is simpler but has a practical inconvenience in the determination of $x$ from the cubic $x^{3}=4^{\kappa}-$ r.

The triangles for which $x=x_{1}, \omega x_{1}, \omega^{2} x_{1}$, are not distinct in ratio of sides.
We shall use this transformation to obtain certain information but shall not discuss it completely.

The extraneous factors $\phi(\phi+r)^{2}$ need notice.

$$
\begin{gathered}
\phi=0 \text { leads to } x=\mathrm{r} y=0 z=0 . \\
a: b: c=\mathrm{r}: 0: 0 \\
K: L: M=\mathrm{r}: 0: 0
\end{gathered}
$$

which is not in general a solution.

$$
\phi=-\mathrm{r} \text { leads to }
$$

$a: b: c=1: \omega: \omega^{2},\left(\omega^{3}=1\right)$ a complex triangle whose angle-bisectors (internal) all vanish and which though a solution of the ratio problem by way of indeterminateness is not a solution of the original problem.

## 4. THE PARAMETRIC FIELDS

Proceeding to the discussion of the equation

$$
F(\sigma, \alpha, \beta)=0
$$

we note that it is a two-parameter equation of order ro irreducible in the domain of rationality constituted by $a \beta$ and their rational functions. For $\alpha$ occurs only to the order r and any reduction in the domain must be of the form

$$
F=M(P u+Q)
$$

where $M, P, Q$ are rational functions of $\beta, \sigma$ alone. Hence the terms containing $\alpha$ and those free from a have a common factor-which is not the case.

The problem can be discussed in three fields:
r) a, $\dot{\beta}$ may be restricted to such values as arise from the assignment of real quantities as the lengths of the angle-bisectors. This may be called the field of real angle-bisectors;
2) $a, \beta$ may be any real quantities;
3) $\alpha, \beta$ may be any algebraic quantities.

In connection with the first field we have a part of, and in connection with the second field the whole of, a real surface which in virtue of the fact that $\alpha$ is single-valued may be considered as a geographical surface and is properly represented by a set of a contour curves covering the whole $\beta, \sigma$ plane.

In connection with the third field we have a "variété" in four dimensions to use the phrase of $M M$. Picard and Simart and in accordance with their theory since the equation is the result of elimination and of numerical genus zero and geometric genus zero, can be birationally transformed into the totality of point pairs of two Neumann spheres. This is accomplished in this case by means of the parameters $\phi, \tau$ in terms of which $\sigma, \alpha, \beta$ are rationally expressed (io) and which are themselves rational in $\sigma, \alpha, \beta$, viz.:

$$
\phi=\sigma-\mathbf{I}, \tau=\frac{\sigma^{3}}{\beta} .
$$

In the field of real angle-bisectors, the quantities $k, l, m$ are necessarily positive and $\alpha \beta$ are positive and restricted to a certain region of the real $\alpha \beta$ plane.


In fact, $k, l, m$ are the roots of the equation

$$
r u^{3}-q u^{2}+p u-\mathrm{x}=0,
$$

and $p, q, r$ are positive.
The discriminant expressed in terms of $\alpha, \beta$ is

$$
\begin{equation*}
\frac{\beta^{2}(\alpha-4)+2 \alpha^{2}(9-2 \alpha) \beta-27 \alpha^{3}}{\alpha^{3} \beta^{2}} \tag{15}
\end{equation*}
$$

which must be positive for real positive roots.
The quartic curve along which the numerator vanishes will be denoted shortly by

$$
D(\alpha, \beta)=0
$$

It has a cusp of the first species at $(0,0)$ where the tangent is $\beta=0$, a cusp of the first species at ( 3,27 ) where the tangent is parallel to $27^{\alpha}=\beta$; asymptotes $\alpha=4$ and $\beta=-\frac{27}{4}$; a parabolic asymptote $\beta=4^{\alpha^{2}}$; and a real finite inflexion without apparent significance in the problenı.

In making a diagram for psychologic reasons connected with the single valued property of $\alpha$, we reverse the usual alphabetic order and take $\alpha$ vertical. The region inside the cusp in the first quadrant is the field of real angle-bisectors. Inside the hyperbolic branch in the second quadrant and between the $\beta$ axis and the curve in the fourth the discriminant is also positive, but as either $a$ or $\beta$ is negative the regions represent real squares of angle-bisectors, some of the set being negative. The curve $D(\alpha, \beta)=0$ in a proper sense and the axes in an improper sense are loci of equal angle-bisectors (Fig. r).

## 5. TWO SPECIAL CASES

At this stage it is convenient to solve some special cases.
First the case of equal bisectors; $K=L=M \quad a=3 \quad \beta=27$.
On writing $\sigma=3 s, F(\sigma, \alpha, \beta)=0$ becomes

$$
16 s^{10}-120 s^{8}+56 s^{7}+297 s^{6}-282 s^{5}-173 s^{4}+348 s^{3}-177 s^{2}+38 s-3=0
$$

which reduces to

$$
(2 s-3)(s-1)^{3}\left(2 s^{2}+3 s-1\right)^{3}=0 .
$$

$s=\frac{3}{2}$ gives the equilateral triangle.
$s=1$ gives $\kappa=\frac{1}{2} z=\infty$ (using the $\kappa, z$ chain) the sides being $a: b: c:: 1: \infty: \infty$; on the scale $a+b+c=1 \quad a: b: c:: \frac{1}{2}: m:-m+\frac{1}{2}$ where $m$ is an infinite quantity.

This is the limit of a triangle with very small base and very long sides when the vertex is brought into line with the base.

Since $c$ is negative the $B$ and $A$ bisectors are external.
This triangle is a constantly occurring triviality.

$$
s=\frac{-3-\sqrt{17}}{4} \text { gives } a: b: c::-1: \frac{5+\sqrt{17}}{4}: \frac{5+11 \overline{1}}{4}
$$

an isosceles triangle with two external bisectors equal to one internal. The angle $A$ is approximately $25^{\circ} 20^{\prime}$.

$$
s=\frac{-3+\sqrt{17}}{4} \text { gives } a: b: c::-\mathrm{I}: \frac{5-\sqrt{17}}{4}: \frac{5-\sqrt{17}}{4}
$$

which are impossible in the sense that $|b|+|c|<|a|$.
As the class will need frequent discussion we reserve the word "impossible" for this exact meaning.

The case $\alpha=4 \beta=54$; the point in the ( $\alpha, \beta$ ) plane being the intersection of the asymptote of $D(\alpha, \beta)=0$ with the curve.

$$
K: L: M:: 2: 2: \Sigma
$$

$\sigma=0$ is a root by inspection. Writing $\sigma=3 s$ as before the remaining terms reduce to

$$
\begin{aligned}
& \quad\left(2 s^{3}+2 s^{2}-7 s+2\right)^{2}(s-2)\left(s^{2}-5\right)=0 . \\
& s=2 \text { gives } \phi=5, \tau=4, y=2 z=\infty .
\end{aligned}
$$

This is the same infinite triangle as in the case $\alpha=3 \beta=27$, a different path of approach to the limit being involved in the different ratio of bisectors.

Using the magnitude of $\sigma$ as an index this root is (2).

$$
s=0 \quad \phi=-1 \quad \tau=0 \quad y=\infty \quad z=\infty \quad \frac{y}{z}=0
$$

This is the limit of $a: b: c:: m: m:-2 m+1$ when $m=\infty$.
The root is ( 7 ).
$s=\sqrt{5}$ gives $a: b: c:: 2: \sqrt{5}-1: \sqrt{5}-1$ the obtuse-angled triangle which occurs in Euclid's construction of a regular decagon. The root is ( 1 ).
$s=-\sqrt{5} \quad a: b: c::-2: \sqrt{5}+1: \sqrt{5}+1$ the acute-angled triangle of the same context. The root is (8).
The cubic factor is irreducible and has three real roots which are to be counted twice each. Approximations are

$$
\begin{array}{lrr}
s=\mathbf{1} .2098 & a: b: c:: \mathbf{1} .947: .584:-\mathbf{1} .53 \mathrm{I} \\
s=.3259 & & \mathrm{I} .247: .045:-.202  \tag{5}\\
s=-2.5357 & & .665: .559:-.233 .
\end{array}
$$

(9) ( r 0$)$

The triangle corresponding to (5) (6) is impossible.

## 6. REALITY OF THE SOLUTIONS

Returning to the general equation $F(\sigma, a, \beta)=0$ it is first to be remarked that as $\sigma$ is a symmetric function of $a, b, c$ only real values of $\sigma$ can lead to real triangles; also that real values of $\sigma$, provided that $\alpha \beta$ are within the field of real angle-bisectors, always lead to real triangles. For real $\sigma$ and real $\beta$ involve real $\phi$ and $\tau$ and so real $y$ and $z$. Hence, $a, b, c$, are either real or one is real and the other two are complex conjugates. But if $a=b, K=L$ and $D(\alpha, \beta)$ vanishes. Now the region of the $u, \beta$ plane where $K, L, M$ are real is finitely scparated from the regions where the squares of these quantities are real and in part negative, the intervening regions having some of the squares complex. It nust be inferred then that real sides and real bisectors are only found together in the single region inside the cusp of $D(\alpha \beta)=0$, this curve being discriminantal for both sets of quantities; and that a real $\sigma$ inside this region means a real triangle.

It is proper to note however that this argument leaves open the possibilities that (I) other triangles besides the isosceles class occur on the locus $D=0$; (2) complex $\sigma$ 's and complex triangles may occur inside the cusp; (3) real sides may occur outside this region, namely with negative squares of bisectors; (4) real triangles inside the cusp may be impossible triangles.

Of these propositions (r) (3) (4) are affirmed and (2) is denied by the subsequent developments.

## 7. TIIE ELEMENTARY THEORY OF EQUATIONS APPLIED

The next undertaking is to use as far as possible the elementary theory of equations in separating the roots, determining their reality, and classifying the triangles connected with them.

On account of the labor involved in some of the operations (such as determining the set of Sturm's functions) being prohibitive, a complete discussion cannot be had by this method, but the results as far as they go are.valuable in themselves and as a corroboration of the subsequent treatment of the problem.

We consider first the functions

$$
\begin{align*}
& \eta_{1}=\sigma^{3}-\beta \sigma+\beta  \tag{16}\\
& \eta_{2}=\sigma^{3}-\beta \sigma+2 \beta
\end{align*}
$$

Using the $\kappa, z$ chain

$$
\begin{equation*}
x^{3}=-\frac{\sigma}{\eta_{1}} \quad z=\frac{\beta(\sigma-1)}{4 \eta_{x}} \tag{17}
\end{equation*}
$$

Assuming $\alpha \beta$ to lie within the real angle-bisector field

$$
a \geq_{3} \quad \beta \geq{ }_{27} ; D(a, \beta)>0
$$

a change of sign of $\eta_{1}, \eta_{2}$ involves a change of sign of $x$ and $z$, and so by means of $\eta_{1}, \eta_{2}$ the roots are classified with respect to their connected triangles.
$\eta_{1}=0$ and $\eta_{2}=0$ have each three real roots, one negative, for $\beta \geq{ }_{27}$.
Considered as curves in the $\sigma, \eta$ plane the families for varying $\beta$ have common properties and one case may be taken as a type. The graph shows $\eta_{1}, \eta_{2}$ for $\beta=54$ and also $F(\sigma, 4,54)$ (Fig. 2).


By reducing $F(\sigma u \beta)$ to the second order by means of $\eta_{1}=0$ and $\eta_{2}=0$ respectively it appears that for the roots of the $\eta$ 's, for $\sigma=0$, and $\sigma=\mathrm{I}$, provided that $\alpha \geq_{3} \beta \geq{ }_{27}, F$ has the following signs:

$$
\begin{array}{cccccccc}
A & B & \circ & \sigma=1 & C & D & E & F \\
+ & + & \pm & + & + & + & - & +
\end{array}
$$

The sign at 0 is + if $a>4$ and - if $a<4$.
$F=0$ has then at least one real positive root between $E$ and $F$.
Since here $x, z$, and hence $y\left(=\left[x^{3}+8 z+\mathbf{r}\right] 4 x\right)$, are positive the problem has a real solution to be interpreted as referring to three internal angle-bisectors.

The signs of $x, z$ in the intervals $-\infty$ to $A, A$ to $B$, etc., are

$$
x-{ }^{A}-\left.\right|^{B}+0\left|-\left.\right|^{\sigma=1}-\left.\right|^{C}+\left.\right|^{D}+\left.\right|^{D}+\left.\right|^{E}+\right|^{F}+
$$

The intervals $C D$ and $E F$ are the only ones in which such an internal solution can occur. No conclusion can be drawn as to $C D$ for $F(\sigma)$ is positive at both $C$ and $D$.

The greater root of $\eta_{2}$ is a superior limit for roots of $F$. A superior limit not so close but more convenient is furnished by $\gamma \beta$.

We next proceed to transform the equation so as to find the number of real roots for intervals in which
(I) $\eta_{1} \eta_{2}$ are both positive,
(2) $\eta_{\mathrm{I}} \eta_{2}$ are negative and positive respectively,
(3) $\eta_{1} \eta_{2}$ are both negative.

For arithmetic convenience we begin by climinating $\sigma$ from $F(\sigma a \beta)=0$ and $\sigma^{3}=\beta(\sigma-\lambda)$. $F$ is first reduced to a quadratic; that is a function $M$ rational and integral in $\sigma, \alpha, \beta$ is determined such that

$$
F(\sigma a \beta)-M\left[\sigma^{3}-\beta(\sigma-\lambda)\right]=N
$$

where $N$ is of the second order in $\sigma$.
This is conveniently done in a tentative fashion and no extraneities are introduced, for the equivalent end term process is

$$
\begin{array}{c|c|c}
\mathrm{I} & F(\sigma \alpha \beta) & 0 \\
-M & \sigma^{3}-\beta(\sigma-\lambda) & \mathrm{I}
\end{array}
$$

and has the determinant of multipliers I .
Thence the result is obtained by substituting in the general eliminant of a cubic and quadratic form. To use the end term process again would introduce an extraneous factor.

The arithmetic is simplified by using $\pi=\lambda-\mathrm{I}$ instead of $\lambda$.
The result arranged as a polynomial in $\pi$ is

|  | $\pi^{20}$ | $\pi$ | $\pi^{8}$ | $\pi{ }^{2}$ | $\pi^{6}$ | $\pi s$ | $\pi 4$ | $\pi{ }^{3}$ | $\pi^{2}$ | $\pi$ | 1 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\alpha \beta^{3}$. | .... | . . . | ... | . . . |  |  | $-16$ | $\bigcirc$ | - | - | $\bigcirc$ |  |
| $\beta^{3}$. | ... | . . . | . . . | . . . |  |  | 64 | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ |  |
| $\alpha^{3} \beta^{2}$. | $\ldots$ | . . . | ... | . . . | 81 | $-36$ | 22 | -4 | I | - | - |  |
| $\alpha^{2} \beta^{2}$. | . . . | . . . . | . . . . | . . . . | $-480$ | 260 | $-76$ | 12 | -4 | - | $\bigcirc$ |  |
| $\alpha \beta^{2}$ | $\ldots$ | . . . |  | . . . . | 768 | $-512$ | $-256$ |  | - | - | 0 | (18) |
| $\beta^{2} .$ |  |  |  |  |  |  |  | 64 | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ |  |
| $\alpha^{3} \beta$. |  |  | -1152 | 800 | 968 | $-354$ | 228 | - | 4 | 2 | 0 |  |
|  |  |  | 3072 |  |  | 1760 |  | $-140$ | I2 | -4 | 0 |  |
| $a^{3}$ | 4096 | $-2048$ | -5376 | 3328 | 1648 | -1224 | -147 | 168 | -6 | -8 | 1 |  |

For the investigation of region (2) where the values of $\lambda$ range from $I$ to 2 we write

$$
\rho=\frac{I-\pi}{\pi}=\frac{2-\lambda}{\lambda-I} .
$$

The equation in $\rho$ will then have positive roots for roots of $F(\sigma)$ in the region (2) that is between the curves $\eta_{1}=0$ and $\eta_{2}=0$.

The equation in $\rho$ is:

|  | $\mathrm{p}^{\text {ro }}$ | $\rho^{s}$ | $\rho^{\prime \prime}$ | $\rho^{7}$ | $\rho^{6}$ | $\rho^{s}$ | $\rho 4$ | $\rho^{3}$ | $\rho^{2}$ | $\rho$ | 1 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\alpha \beta^{3}$. | ... |  |  |  | -16 | -96 | -240 | $-320$ | -240 | -96 | -16 |  |
| $\beta 3$. |  |  |  | . . . | 64 | 384 | 960 | I 280 | 960 | 384 | 64 |  |
| $\alpha^{3} \beta^{3}$. |  |  | 1 | 4 | 22 | 68 | 161 | 320 | 400 | 256 | 64 |  |
| $\alpha^{2} \beta^{2}$. | . . . | . | 4 | -20 | -104 | -168 | -180 | -644 | -1280 | -1024 | -288 | (19) |
| $\alpha \beta^{\beta}$. |  |  |  | $-16$ | $-368$ | $-2384$ | -6192 | $-7728$ | $-4688$ | -1136 | $-16$ |  |
| $\beta^{2}$. |  |  | $\ldots$ | 64 | 448 | I 344 | 2240 | 2240 | 1344 | 448 | 64 |  |
| $\alpha^{3} \beta$ |  | 2 | 22 | 104 | 508 | 1490 | 3150 | 6084 | 7120 | 3616 | 496 |  |
| $\alpha^{2} \beta$ |  | -4 | -24 | - 188 | $-592$ | 1316 | 7240 | 6972 | $-832$ | $-2592$ | -288 |  |
| $\alpha^{3}$ | - I | 2 | $-33$ | -48 | 399 | 330 | -2015 | $-268$ | 3168 | $-2160$ | 432 |  |

The transformation is of course effected in the coefficients of the powers and products of $a, \beta$, separately.

Proceeding with the determination of signs of the coefficients:
$\left[\rho^{10}\right] a^{3}$ is positive in the region,
$\left[\rho^{\rho}\right] 2 a^{2}(a+a \beta-2 \beta)$ is positive,
$\left[\rho^{\beta}\right]$ and $\left[\rho^{p}\right]$ vary and are reserved for later discussion.
[ $\rho^{6}$ ] This coefficient can be arranged as

$$
-16(\beta+1) D+a^{3}\left(-\beta^{2}+12 \beta-33\right)+\beta^{2}\left(-4 \mathfrak{1}^{3}+184 a^{2}-35^{2 a}+384\right)-304 a^{2} \beta
$$

where $D$ is written for $D(a \beta)$ which is positive in the region.
Since $\alpha \geq_{3}$ and $\beta \geq_{27}$ this is negative throughout the region.
[ $\rho^{5}$ ] can be arranged as

$$
-96 \beta D-\beta^{2}\left(315^{a^{3}}-1560 a^{2}+2384^{a}-1344\right)-\beta\left(1102 a^{3}-1316 a^{2}\right)+a^{3}\left(-\beta^{2}+330\right)
$$

which is negative.
[ $\rho^{4}$ ] is

$$
-240 \beta D-\beta^{2}\left(799^{3}-4140 \alpha^{2}+6192 a-2240\right)-\beta\left(3330 a^{3}-7240 a^{2}\right)-2015 a^{3}
$$

which is negative.
$\left[\rho^{3}\right]$ is

$$
-320 \beta D-\beta^{2}\left(960 a^{3}-5116 a^{2}+7728 a-2240\right)-\beta\left(2556 a^{3}-6972 a^{2}\right)-268 a^{3}
$$

which is negative.
$\left[\rho^{2}\right]$ is

$$
-240(\beta+1) D-128(a-3) \beta^{2}-(4320 \beta-3488 a) a \beta-320 a^{3} \beta-3312 a^{3}
$$

which is negative.
[ $p$ ] is

$$
(-96 \beta+80) D+64(a-3) \beta\left[\beta\left(-a^{2}+10 \alpha-3^{2}\right)+\alpha^{2}(-\beta+21)\right]
$$

which is negative.
[ $\rho^{\circ}$ ] is

$$
-16(\beta+1) D
$$

which is negative.
Returning to the eighth and seventh powers,

$$
\left[\rho^{2}\right]-4\left[\rho^{8}\right]=4\left[-4 D+4^{\alpha^{2}} \beta(-a+3)+4^{\alpha^{2} \beta^{2}}(-\beta+10)-87^{a^{3}}\right]
$$

which is negative.
Hence if $\left[\rho^{8}\right]$ is negative $\left[\rho^{7}\right]$ is also.
Of the eleven signs either the first two, three, or four are positive and all others negative throughout the region and $F(\rho)=0$ has one and only one positive root.

From the last result it follows that $F(\sigma, a, \beta)=0$ has one root and only one root in the intervals $A \cdots B, \mathrm{C} \cdots D, E \cdots F$ and since there is one root in $E \cdots F$ which leads to a triangle with positive sides and therefore internal bisectors in the given ratios and since all such roots fall in these intervals there is one and only one solution of the original geometrical problem for every set of real quantities assigned as the lengths of the internal angle-bisectors. Since this root is greater than 3 and less than $\gamma^{\prime} \beta$ or less than the greatest root of $\sigma^{3}-\beta \sigma+2 \beta=0$ and no other root occurs in the interval there can be no trouble in separating the root. The sides are to be determined from $\sigma$ by rational operations and solution of the cubic $\left(t^{3}-t^{2}+t y-z=0\right)$.

In a certain narrow sense this is a solution of the problem, but it is necessary to inquire more closely as to the character of the irrationality involved and as to the necessarily introduced extraneities and their connection.

Proceeding with the method in hand we write $\psi=\lambda-2$.
The equation in $\psi$ will have positive roots corresponding to roots of $F=0$ in the intervals $-\infty$ to $A$ and $D$ to $E$. As however the least root of $\eta_{2}=0$ is an inferior limit the interval $D \cdots E$ is the only one in question.

In this case the labor required is not justified by the results to be expected, and it may merely be noted that the coefficients of $\psi^{10}$ and $\psi^{9}$ are positive while the constant term is $-16(\beta+1) D$.

From which at least one root results.
By writing $\theta=-\frac{1}{\pi}$ we have positive $\theta$ 's associated with roots of $F=0$ in the intervals $B \cdots C$ and $F \cdots+\infty$ but as the greatest root of $\eta_{1}=0$ is an upper limit the interval $B \cdots C$ is alone possible.

Here taking the coefficients from the polynomial in $\pi$ with the proper changes in order and sign . -
[ $\theta^{10}$ ] is $a^{3}$ and is positive,
[ $\theta^{9}$ ] is $8 a^{3}+4 a^{2} \beta-2 a^{3} \beta$ and is always negative,
$\left[\theta^{8}\right]$ is $\alpha^{3} \beta^{2}-4 \alpha^{2} \beta^{2}+4 \alpha^{3} \beta+12 \alpha^{2} \beta-6 a^{3}$.
This varies in sign, being negative at $(3,27)$ and positive at $(4,54)$.
$[\theta 7]$ is $4\left[\alpha^{3} \beta^{2}-3 \alpha^{2} \beta^{2}+4 D+(16 a-37) \alpha^{2} \beta+66 a^{3}\right]$ and is positive.
$\left[\theta^{6}\right]$ is $-16 \beta D+i^{2}\left(-42 \alpha^{3}+212 \alpha^{2}-256 \alpha\right)+\alpha^{2} \beta(-294 \alpha+388)-147^{3}$.
The coefficient of $\beta^{2}$ has the value 6 for $a=3$ but rapidly diminishes, the derivative being -II8, and has negative values for $\alpha \geq 3.05 \ldots$, while $\beta$ is very near 27 if this coefficient is positive. Hence the whole expression is easily seen to be negative.
[ $\theta_{5}$ ] varies in the region.
$\left[\theta^{4}\right]$ is $\left(81 \alpha^{2}-480 \alpha+768\right) a \beta^{2}+(968 a-2816) u^{2} \beta+1648 \alpha^{3}$ and positive.
$\left[\theta^{3}\right]$ is $32 \alpha^{2}\left(-25^{\alpha} \beta+80 \beta-104 \alpha\right)$.
The coefficient of $\beta$ is negative for $\alpha>3.2$ when $\beta<40$.
Hence the whole expression is less than - $112\left(32 \alpha^{2}\right)$.

$$
\left[\theta^{2}\right] \text { is } 384^{\alpha^{2}}\left(-3^{\alpha} \beta+8 \beta-14^{\alpha}\right) \text { and negative. }
$$

The remaining coefficients are positive.
The signs then run in the interval

$$
+- \pm+- \pm+--++
$$

The variations of $\left[\theta^{8}\right]$ and $\left[\theta^{3}\right]$ have then no effect and there may be always six roots in the interval.

With these results the practical utility of the classical theory seems to end. The direct evaluation of the discriminant in Bezout's form is impracticable, the elements of the ninth order determinant involving $\alpha$ and $\beta$ in polynomials of the fourth order with coefficients of six figures. Still less is it to be expected that a successful attack could be made on Sturm's functions.

Recourse must then be had to special methods applicable to this problem and to the powerful general method of following the transformations used in the elimination and interpreting them.

## 8. REALITY OF THE ROOTS FOR REAL ANGLE-BISECTORS

If the bisectors are real $k, l, m$ are positive. Write $t=(a+b+c) a b c$ and $y_{\mathrm{r}}=\frac{\mathrm{I}}{k t}, y_{2}=\frac{\mathrm{I}}{l t}$, $y_{3}=\frac{\mathrm{I}}{m t}$, the internal formulas become

$$
\begin{equation*}
y=\frac{1-2 x}{x(1-x)^{2}} \text { with } x=a \text { for } y_{\mathrm{r}} ; x=b \text { for } y_{2} ; x=c \text { for } y_{3} \tag{20}
\end{equation*}
$$

The curve represented by this equation has the line $x=1$ for an asymptote with a cusp at infinity. The $x$ axis is also an asymptote with an inflexion at infinity, the curve approaching with negative $y$. The $y$ axis is an ordinary asymptote. The curve meets the $x$ axis also at $x=\frac{1}{2}$ (Fig. 3).


Consider first $t$ positive and therefore $y$ positive.
For any $t$ the lines $y_{1}=\frac{1}{k t} ; y_{2}=\frac{1}{l t} ; y_{3}=\frac{1}{m t}$ each cut the curve in one real point. The sum of the abscissas is $\frac{3}{2}$ when $t=\infty$ and is o when $t=0$. Since the curve monotonously approaches the $y$ axis as $t$ decreases the value I is reached once and only once by $a+b+c$; hence there is always one and only one solution for a positive $t$.

The original problem restricted to internal bisectors has a unique solution, for it is evident that $a b c$ and $t$ must all be positive for this interpretation. As to the possibility of the values $a b c$ we note that since the perimeter is $I$ and each side is less than $\frac{1}{2}$ any two must be together greater than the third.

Considering negative $t$ 's each of the lines $y_{\mathrm{s}}=\frac{1}{k t}$, etc., cuts the curve in three real points.
Assuming that $k<l<m$ we call the abscissas

$$
a_{1}<a_{2}<a_{3} ; \quad b_{1}<b_{2}<b_{3} ; \quad c_{1}<c_{2}<c_{3} .
$$

The product of 27 factors

$$
P\left[\mathrm{r}-\left(a_{i}+b_{j}+c_{k}\right)\right] \quad i, j, k=\mathrm{1}, 2,3
$$

is symmetric in the $a^{\prime} s, b$ 's, and $c$ 's separately and can be expressed in terms of $k, l, m, t$.
This product vanishes for solutions and only for solutions.
From the other eliminations we know that there are in general ten finite values of $t$ satisfying $P=0$. We have seen that one and only one is positive.

For negative $t$ 's we consider the 27 combinations $a_{i}+b_{j}+c_{k}$ denoting them shortly by $(i, j, k)$ understanding that the indices in the order written refer to $a, b, c$.

Some of the 27 are excluded as being continuously greater or less than 1 . These are

$$
\begin{aligned}
& (1,1,1)<0 \\
& (2,2,2)>\frac{3}{2} \\
& (2,2,3),(2,3,2),(3,2,2)>2 \\
& (2,3,3),(3,2,3),(3,3,2)>\frac{5}{2} \\
& (3,3,3)>3 \\
& (1,1,2),(1,2,1),(2,1,1)<1
\end{aligned}
$$

There remain 15 arrangements which may possibly yield solutions.
Taking the set $(1,2,2) a_{1}+b_{2}+c_{2}=2$ for $t=0, y=\infty$

$$
=-\infty \text { for } t=\infty, y=0
$$

Hence at least one real root, for the abscissas all decrease monotonously with increasing $t$, and are finite and continuous in the interval $0>t>-\infty$.

Similarly ( $2,1,2$ ) and ( $2,2,1$ ) each yield at least one real root.
Of the six permutations of (123) three can be excluded, namely:

$$
\begin{aligned}
& (1,2,3):\left(a_{1}+b_{2}+c_{3}\right)>\left(c_{1}+c_{2}+c_{3}\right)>2 \\
& (1,3,2):\left(a_{1}+b_{3}+c_{2}\right)>\left(b_{1}+b_{2}+b_{3}\right)+\left(c_{2}-b_{2}\right)>2-\frac{1}{2}>1 \\
& (2,1,3):\left(a_{2}+b_{1}+c_{3}\right)>\left(c_{1}+c_{2}+c_{3}\right)>2
\end{aligned}
$$

For $(2,3,1) a_{2}+b_{3}+c_{1}=2$ for $t=0$; for $t=-\infty, y=-0$, the equation

$$
x^{3}-2 x^{2}+x(\mathrm{I}+2 k t)-k t=0
$$

has one root $\frac{1}{2}$ and the others approach infinity with $\pm 1 \overline{-2 k t}$.
As $k<l<m, a_{2}+b_{3}+c_{1}$ approaches negative infinity with $\frac{1}{2}+\sqrt{-2 l t}-\sqrt{-2 m} t$.
Hence (23I) yields at least one real root.
In a similar manner it can be shown that (312) and (321) each give at least one real root.
In the set of permutations of (II3) we have in the case (131) $a_{1}+b_{3}+c_{1}=1$ for $t=0$ and is negatively infinite when $t$ is.

To determine whether it ever exceeds i in the range we express the roots of

$$
\begin{equation*}
x^{3}-2 x^{2}+(\mathrm{I}+2 k l) x-k l=0 \tag{2I}
\end{equation*}
$$

in terms of $k t$ near $x=0, t=0$ and $x=1, t=0$.

$$
\begin{align*}
& x_{1}=k t-k^{3} t^{3}-2 k^{4} t^{4} \ldots  \tag{22}\\
& x_{2}=1-1 \overline{-k t}+\frac{k t}{2}+\cdots \\
& x_{3}=1+1 \overline{-k t}-\frac{k t}{2}+\cdots \\
& a_{1}+b_{3}+c_{1}=k t+\mathrm{I}+1 \overline{-l t}+m t+\ldots
\end{align*}
$$

Since in the limit as $l$ approaches -o the square root term exceeds in absolute value the sum of the terms of the first order, (13I) is greater than one in the region of $t=-0$, and so gives at least one real root. Similarly ( 3 ri) gives a real root at least.

The treatment is not sufficient for (II3) as this may not approach minus infinity with $t$. Reserving this case and taking the $1,3,3$ set

$$
\begin{aligned}
& \text { (133): } a_{1}+b_{1}+c_{3}>c_{1}+b_{3}+c_{3}>c_{1}+c_{2}+c_{3}>2 \\
& \text { (3I3): } a_{3}+b_{1}+c_{3}>b_{1}+b_{2}+b_{3}>2
\end{aligned}
$$

(331) requires a more detailed examination and is to be considered in connection with (113).

$$
\begin{aligned}
& \text { (1 п ) } a_{1}+b_{1}+c_{3}=1 \text { when } t=0 \\
& \left(33 \text { I) } a_{3}+b_{3}+c_{1}=2 \text { when } t=0 .\right.
\end{aligned}
$$

When $t$ approaches minus infinity (I I 3) approaches $1 \overline{-2 t}\left[-1 / k-1^{\prime} l+1 / m\right]$ while (33 1 ) approaches the same quantity with reversed sign.

So one and only one of the two combinations gives at least one real root in the interval.
In terms of the fundamental parameters $1 k+1 / \gg m$ reads $\alpha<4$ and in this case (II3) has the root, but if $\alpha>4$ (33I) has it.

The boundary case $\alpha=4$ needs special mention as in this event the first approximation is indeterminate.

Consider the limit $\begin{gathered}k=k_{\mathrm{x}} \\ \left.t=-\sqrt{\prime}-\mathrm{l}^{\prime} l+\mathrm{V} / m\right)(\overline{-2 l}) \\ t=-\infty\end{gathered}$
where $k_{\mathrm{x}}$ is such a number that $\mathrm{V}^{\prime} k_{\mathrm{x}}+\mathrm{I}^{\prime} l-V^{\prime} m=0$.
We have

$$
\begin{aligned}
& \text { Limit }=\text { limit } \quad \frac{\left(-\sqrt{k_{1}+x-1}+1+m\right) V^{\prime} 2}{V^{\prime} y} \\
& =\frac{\sqrt{2}=0}{\sqrt{k_{1}}} \operatorname{limit} \frac{x}{\sqrt{x} y} .
\end{aligned}
$$

If then the path of approach be along the parabola $2 x^{2}=k_{1} y$ the limit is 1 . So we may consider that (II3) has a root at the limit. Or by approaching on the other side (33I) has a root here.

For $a=4$ the original equation has a root $\sigma=0$ which leads by the chain of equations to just such an infinite triangle as is here in question.

In every case then there are at least ten real roots for finite values of $t$, and as we know that there are only ten such roots there must be one in each of the classes specified.

The arrangement of the bisectors in the triangle should be noted. For the internal case the magnitudes of the bisectors and the sides which they meet are in reversed order.

For the case (23r) the internal bisector is $M$ since $c_{1}$ is negative.
The least bisector is internal but no further statement can be made as to absolute magnitudes, for if $t$ is large $c_{1}$ may in absolute value exceed $b_{3}$.

For (312) the medium bisector is internal, for (32I) the least. So (122) has the greatest internal, (212) the medium, and (221) the least.

The set I, I, 3 are all impossible triangles (see \& II) and the words internal, external are without a proper meaning, but we may say, in ( $\mathrm{r}_{3}$ ) the medium bisector is associated with the side that is unique in sign (b).

In (3II) the greatest and in (II3) the least is so associated. If $\alpha>4$ (331) gives a possible triangle with the least bisector internal. The want of symmetry of the arrangement by which the least bisector is "internal" in five analytic cases out of nine and in four real cases out of eight (or in three out of seven) is to be noticed.

## 9. A PRACTICAL METHOD FOR APPROXIMATE SOLUTIONS

Following the plan of the last section we have the three equations (2I) with $k, l, m$ given and a particular set of three roots, say $a_{2}, b_{3}, c_{1}$ whose sum is to be unity. For the sets $(3,2,1)$ $(2,3, I)(3, I, 2)(I, 2,2)(2, I, 2)(2,2, I)$ it is easy to show that the sum decreases monotonously as $t$ increases. For the other negative sets this is not so, but the solution occurs on such a slope which is unique. By trial we may find values of $t$ between which the solution lies.

After the first operation the convergence is satisfactory, the curve 'sum $=F(t)$ ' being flat in character. The result carries with it the class of triangle.

Conceivably this process could be automatically carried out by a machine.
Parallel bars set at distances proportional to $k, l, m$ and kept parallel by a parallelogram linkage, intersect fixed curves patterned to equation (20). A stretched string with its length greater than the sum in question by a constant would reach a mark as the solution is attained.

It is to be noted that this method avoids the elimination.

IO. MULTIPLE POINTS
Considering the fundamental equations in the form:

$$
\begin{align*}
& F_{a} \equiv a^{3}-2 a^{2}+(\mathrm{I}+2 k t) a-k t=0 \\
& F_{b}=0, F_{c}=0 ; a+b+c=\mathrm{I} \tag{2I}
\end{align*}
$$

If $a, b, c, k, l, m, t$ are a consistent set of values, for neighboring values we have:

$$
\begin{equation*}
\frac{\partial F}{\partial a} \delta a+\frac{\partial F}{\partial t} \delta t+\frac{\partial F}{\partial k} \delta k=0 \tag{23}
\end{equation*}
$$

and two similar equations. If the point is an ordinary point at which moreover none of the partial derivatives $\frac{\partial F}{\partial a} \frac{\partial F}{\partial b} \frac{\partial F}{\partial_{c}}$ vanish, we have three equations of the form:

$$
\begin{align*}
& \delta a=A \delta t+B \delta k  \tag{24}\\
& \delta b=C \delta t+D \delta l \\
& \delta c=E \delta t+F \delta m
\end{align*}
$$

First suppose $A+C+E \neq 0$ and solve

$$
0=\delta a+\delta b+\delta c=(A+C+E) \delta t+B \delta k+D \delta l+F \delta m \text {. }
$$

This gives one value of $\delta t$ and so one solution in the neighborhood $(\delta k, \delta l, \delta m)$ of $(k, l, m)$, that is of $(a, \beta)$.

Secondly if

$$
\begin{equation*}
A+C+E \equiv \Sigma^{a(a-1)(2 a-1)} \frac{4 a^{2}-3 a+1}{}=0 \tag{25}
\end{equation*}
$$

there is a double point in the neighborhood.
Setting out the complete increment equations after elimination of $k, l, m$ :

$$
\begin{align*}
\delta a & =\frac{-(a-1)(2 a-1) a}{4 a^{2}-3 a+1} \delta t+\frac{(2 a-1) 2 t^{2}}{(a-1)\left(4 a^{2}-3 a+1\right)} \delta k \\
& +\frac{2 t(2 a-1)}{(a-1)\left(4 a^{2}-3 a+1\right)} \delta a \delta k-\frac{2 a(a-1)}{t\left(4 a^{2}-3 a+1\right)} \delta a \delta t \\
& +\frac{(2 a-1)^{2}}{(a-1)\left(4 a^{2}-3 a+1\right)} \delta t \delta k-\frac{2(3 a-2)(2 a-1)}{(a-1)\left(4 a^{2}-3 a+1\right)} \delta a^{2}  \tag{26}\\
& -\frac{6(2 a-1)}{(a-1)\left(4 a^{2}-3 a+1\right)} \delta a^{3}
\end{align*}
$$

and two similar.
When $\leq \frac{a(a-1)(2 a-1)}{4 a^{2}-3 a+1}=0$ we have by addition an equation of the form:

$$
B \delta k+D \delta l+F \delta m=0
$$

in which unless $t=0$ or $a, b, c$ have certain special values $B, D, F$ are finite both ways.
If $a, \beta$ and $\frac{d a}{d \beta}$ are assigned the ratios $k+\delta k: l+\delta l: m+\delta m$ are given.
Let

$$
\begin{equation*}
\frac{k+\delta k}{k^{\prime}}=\frac{l+\delta l}{l^{\prime}}=\frac{m+\delta m}{m^{\prime}}=p \tag{27}
\end{equation*}
$$

Then

$$
\begin{equation*}
p^{\prime}=\frac{B k+D l+F m}{B k^{\prime}+D l^{\prime}+F m^{\prime}} \delta k=p k^{\prime}-k, \text { etc. } \tag{28}
\end{equation*}
$$

For an ( $\alpha \beta$ ) arbitrarily near a point for which $A+C+E=0$ the terms of the second order on substitution of these values for $\delta k \delta l \delta m$ and the values given by the first approximations for $\delta a \delta b \delta c$ give a quadratic for $\delta t$.

Such points then are double points and the locus a discriminantal locus.
Naming it

$$
\begin{equation*}
T \equiv \mathbf{\Sigma} \frac{a(a-\mathrm{I})(2 a-\mathrm{r})}{4 a^{2}-3 a+\mathrm{I}}=0 \tag{29}
\end{equation*}
$$

we recognize that $T$ must be a factor in the discriminant of $F(a, k ; l, m)=0$ and also of $F(\sigma, a, \beta)=0, F(a, k ; l, m)=0$ being the equation for the side $a$ in terms of $k, l, m(\S$ I5).

A triple-point locus, that is a locus whose intersections with $T=0$ are triple points, can be obtained by forming the second derivatives of $a, b, c$ with respect to $t$ and expressing the condition that their sum vanishes.

In terms of $a, b, c$ this is

$$
\begin{equation*}
S=\Sigma \frac{(a-1)(2 a-1)\left(4 a^{2}-6 a+3\right) a^{3}}{\left(4 a^{2}-3 a+1\right)^{3}}=0 \tag{30}
\end{equation*}
$$

To express this in the $y, z$ plane, however, requires a laborious elimination, and, owing to the practical necessity of multiplying up the denominators the result (which aside from an infinite value of $z$ is of the fourteenth order in $z$ ) is not free from extraneities, in fact for the point

$$
a=\frac{3+v-7}{8}, \quad b=\frac{3-v-7}{8}, \quad c=\frac{1}{4}
$$

both $T$ and $S$ vanish, but the point in fact is only a double point. The finite triple points will be determined ( $\$ 12$ ) by another method.

Multiple points however can occur when $T$ does not vanish. If $k=l, a \neq b$

$$
\begin{equation*}
\delta t=-\frac{(B \delta k+D \delta l+F \delta m)}{A+C+E} \tag{3I}
\end{equation*}
$$

and in this formula interchange of $(\delta k, \delta l)$ in general changes this value but leaves $\alpha+\delta \alpha$, $\beta+\delta \beta$, which are symmetric functions of $k+\delta k, l+\delta l, m+\delta m$ unchanged. Hence $\delta t$ has two values and the locus is discriminantal for $F(\sigma, \alpha, \beta)=0$ but not for $F(a, k ; l, m)=0$.

The finite number of points where $k=l, a \neq b$, and $B=D$ are given by $4 c^{3}+4 c^{2}+c-\mathrm{I}=0$ and form an apparent exception but as they are discrete points on a continuous locus there are still two values of $\delta t$ for points as close to them as we please.

If $k=l, a=b$, then $B=D$ and $\delta t$ has only one value.
Naming the locus $k=l, a \neq b$ as $D_{2}=0$, we see that for values of $\alpha, \beta$ satisfying the ( $\alpha, \beta$ ) equation of the locus the double root recurs three times.

For the first two equations of the system (2I) are identical and if $a_{1}, a_{2}, a_{3}$ are their roots and $a_{1}, a_{2}, c$ are the sides of the triangle $a_{1}+a_{2}+c=1$ and $a_{1}+a_{2}+a_{3}=2$; whence $a_{3}=c+1$ and if we write $k=1$

$$
\begin{equation*}
m=\frac{c(c-\mathrm{I})^{2}}{\mathrm{I}-2 c} \times \frac{\mathrm{I}-2(c+\mathrm{I})}{(c+\mathrm{I}) c^{2}} . \tag{32}
\end{equation*}
$$

Setting aside $c=0$ as not finite, we have a cubic in $c$

$$
2 c^{3}(m-1)+c^{2}(m+3)-m c+1=0
$$

For every $m$, that is for every $a, \beta$ on the locus, there are threc values of $c$ and so three double points. The discriminant of the cubic is

$$
\begin{equation*}
m\left(9 m^{3}+38 m^{2}+9 m+216\right) \tag{33}
\end{equation*}
$$

If $m$ vanishes the double value of $c$ is $I$ and the corresponding triangle has sides in the ratios

$$
1: \frac{\mathbf{I}}{\sqrt{3}}:-\frac{\mathbf{I}}{\sqrt{/ 3}}
$$

This is an eight-point but $\beta$ is infinite and so it is not to be listed in the finite multiple points. For the cubic factor, which is irreducible, there are three four-points, one with a value of $m$ near -5 and two complex values. These are scen later to correspond to intersections of $T=0$ and $D_{2}=0$.

For the locus $k=l, a=b$, the locus of isosceles triangles, to be named $D_{1}=0$, proceeding as above,

$$
\begin{equation*}
m=\frac{8 c^{2}(c-1)^{2}}{(2 c-1)(c+1)^{2}(c-1)} \tag{34}
\end{equation*}
$$

If $c=\mathrm{I}$, then $m={ }_{0}^{0}, t=0, a, \beta$ are indeterminate. The triangle has sides as $\mathrm{I}: 0: 0$.
There remain three values of $c$ for every $m$ leading to isosceles triangles given by

$$
\begin{equation*}
2(m-4) c^{3}+(3 m+8) c^{2}-m=0 \tag{35}
\end{equation*}
$$

The discriminant is

$$
\begin{equation*}
m\left(27 m^{2}+9 m+32\right) \tag{36}
\end{equation*}
$$

For $m=0$ the double value is $c=0$, not finite while the complex values of $m$ lead to two complex triangles in which a part of the isosceles solutions coincide.

These are

$$
a: b: c:: \frac{19 \mp_{1}-15}{94}: \frac{19 \mp_{1}-15}{94}: \frac{28 \pm 1-15}{47} .
$$

The points are only double points in the general problem.
Next are to be considered the points which are singular in the individual equations of the system.

If $\frac{\partial F}{\partial a}=0$ the equations take the form

$$
\begin{align*}
& \delta a^{2}=A^{\prime} \delta t+B^{\prime} \delta k \\
& \delta b=C \delta t+D \delta l  \tag{37}\\
& \delta c=E \delta t+F \delta m
\end{align*}
$$

which give

$$
\begin{equation*}
(C+E)^{2} \delta l^{2}+\left[2(C+E)(D \delta l+F \delta m)-A^{\prime}\right] \delta t+\left[(D \delta l+F \delta m)^{2}-B \delta k\right]=0 \tag{38}
\end{equation*}
$$

Since this quadratic has one root which approaches zero with $\delta k, \delta l, \delta m$ as a rule only an ordinary point exists for the problem. If $A^{\prime}$ vanishes, which happens for $a=1$, one of the values in question, the point is a double point at least, but as $\beta$ is infinite not among the finite set.

If $l=m, b \neq c$ there is a double point as before on the locus $D_{2}=0$.
If $\partial F_{a} / \partial a$ and $\partial F_{b} / \partial b$ both vanish and $a=b, \delta t$ is given by a quartic, three of whose roots approach zero with $\delta k, \delta l, \delta m$. This gives triple points for the triangles

$$
a: b: c:: \frac{3+v-7}{8}: \frac{3+v-7}{8}: \frac{1-1-7}{4}
$$

and the conjugate values which are on $T=0$ and $a: b: c:: \mathrm{I}: \mathrm{I}:-\mathrm{I}$ for which $a=\infty, \beta=\infty$.
If both the partial derivatives vanish and $a \neq b, \delta t$ is given by a quartic two of whose roots approach zero.
'This applies to the triangles

$$
\begin{aligned}
a: b: c & :: \frac{(3+1-7)}{8}: \frac{(3-1-7)}{-8}: \frac{1}{4} \\
& :: \text { г } \\
& :: \frac{(3+1-7)}{8}: \frac{(-5-1-7)}{8} \\
& : \frac{(3-1-7)}{8}: \frac{(-5 .+1-7)}{8}
\end{aligned}
$$

which are double points merely, though they formally satisfy $T=0$ and $S=0$.
Three of the partial derivatives cannot vanish together as $a+b+c=1$ and the only permissible values are $\mathrm{I}, \frac{(3 \pm 1 /-7)}{8}$.

If a partial derivative is infinite a side has the value $\frac{1}{2}$ from which $k$ is infinite and $\beta$.
Finally none of the special values give finite multiple points not on one or other of the discriminantal loci $T=0, D_{2}=0$.

The nature of this investigation naturally leaves a doubt as to its sufficiency and may be regarded as a mere reconnaissance which serves the purpose of gathering material to lighten the labor of a conclusive determination subsequently undertaken ( $\$ \mathrm{I} 2$ ).

## II. THE TRANSFORMATIONS

Beginning with the configuration in the $(a, b, c)$ plane, take as reference triangle an equilateral triangle and as the homogeneity relation $a+b+c=1$. Every point in the plane represents an analytic triangle with unit perimeter. Each such triangle has a sixfold representation corresponding to the permutations of $a, b, c$, the six points having a sixfold central symmetry.

The lines $-a+b+c=0, a-b+c=0, a+b-c=0$ form a proper triangle dividing the plane into compartments. Inside, all the triangles represented have positive sides and are possible triangles. In the regions outside one of the lines the represented triangles are impossible. Outside two of the lines the triangles can be constructed, one side being considered negative, and the bisectors meeting this side being external. The sixth part of the plane shown in Fig. 5 has the regions $\oplus, 0,00,000,5,6,7$, as subdivisions of the region of impossible triangles with real sides while the regions $1,2,3,4,7^{\prime}, 8,9$, 10, include all the real possible triangles, the region I alone representing the real possible triangles with positive sides, that is triangles in the ordinary sense.

In the regions 5, 6, 7, the angle-bisectors are real though the triangles are impossible, while in $\oplus, 0,00,000$, the bisectors are pure imaginary quantities.

Considering in detail the discriminantal curves, we have first

$$
\begin{equation*}
T \equiv \Sigma \frac{a(\mathrm{I}-a)(\mathrm{I}-2 a)}{4 a^{2}-3 a+\mathrm{I}}=0 \tag{20}
\end{equation*}
$$

In the $y, z$ plane the form is

$$
\begin{equation*}
4 y^{2}-20 y z+40 z^{2}-y+z=0 \tag{39}
\end{equation*}
$$

As this is an ellipse every real $(y, z)$ is finite and so every real $(a, b, c)$ is finite and the curve is closed in the $(a, b, c)$ plane. ' It touches the sides of the reference triangle at the mid-points and passes through the vertices perpendicular to the medians.

From the summation form it has no point such that $a, b, c$ are all positive and less than $\frac{1}{2}$.


By writing $c=1-a-b, a+b=2 \xi, a-b=2 \eta$ a sextic is obtained which contains $\eta$ only as $\eta^{2}, \eta^{4}$ from which any number of points are easily obtained and multiplied by the sixfold symmetry.

The curve is a trefoil not entering the reference triangle and crossing the compartments - and 00 in the fundamental region (Figs. 4, 5).

The set of lines $\pi(a-b)=0$ while not discriminantal are so transformed as to lie on $D(\alpha, \beta)$ which is discriminantal.

This locus of isosceles triangles has been named $D_{1}$.
The locus of equal bisectors for scalene triangles $k=l, a \neq b$ has for its equation

$$
\begin{equation*}
2 a b(a+b)-\left(a^{2}+5 a b+b^{2}\right)+2(a+b)-1=0 \tag{40}
\end{equation*}
$$

and is to be taken with two other curves obtained by cyclic interchange to constitute the complete locus

$$
\begin{equation*}
D_{2}(a, b, c)=0 \tag{4I}
\end{equation*}
$$

Each constituent has three real asymptotes

$$
a=\frac{1}{2}, \quad b=\frac{1}{2}, \quad c=\frac{1}{2}
$$

with cusps at the infinite point, and no finite singularity. Two branches touch the sides of the reference triangle at the vertices, and externally.

In the fundamental region (Fig. 5) the various branches separate the regions 5 and 6, Io and 9, 8 and 9, 4 and 3,3 and 2,6 and 7.

Other discriminantal lines are $\pi(a+b)=0$, which leads to $\beta=\infty$ and $\pi(a+b-c)=0$ giving $\beta=0$. The loci corresponding to zero and infinite values of $\alpha$ are too complicated for their utility at this stage and are added later.

In the fundamental region $a+b=0$ separates 00 and $000 ; 4$ and $9 ; 3$ and $8 ; 2$ and $7^{\prime}$. The line $a+b-c=0$ separates I and $\oplus$; 1о and $0 ; 9$ and $00 ; 4$ and 000 .

To the ( $a, b, c$ ) plane and this collection of curves, following the process of the elimination, is now applied the "elementary symmetric function transformation,"

$$
\begin{align*}
a+b+c & =x, \\
a b+b c+c a & =y,  \tag{3}\\
a b c & =z
\end{align*}
$$

Passing to a rectangular system by writing $x=1$ and transforming the various curves the configuration in the $y, z$ plane can be set out. The transformation is point for point between the fundamental region in the $(a, b, c)$ plane and the region within the discriminantal curve in the $(y, z)$ plane. This curve is

$$
\begin{align*}
& D_{1}=(a-b)(b-c)(c-a)  \tag{42}\\
& D_{1}=4 y^{3}-y^{2}-18 y z+27 z^{2}+4 z=0
\end{align*}
$$

in the $(y, z)$ plane.
The region of the $(y, z)$ plane without this curve represents complex points in $(a, b, c)$ the ratios of the "sides" being of the form

$$
a: b: c:: \mathrm{I}: p+q_{V} /-\mathrm{I}: p-q \sqrt{ }-\mathrm{I}
$$



Six such points corresponding to the six permutations of $a, b, c$ are represented by each point in $(y, z)$.

We may say shortly that the transformation brings up from the complex regions those triangles whose sides have ratios which while not real are the roots of a cubic equation with real coefficients (Fig. 6).

The curve $D_{\mathrm{I}}(y, z)=0$ is a semi-cubical parabola and can be written as

$$
\begin{equation*}
4(3 y-1)^{3}+(9 y-27 z-2)^{2}=0 \tag{43}
\end{equation*}
$$

The cusp ( $y=\frac{1}{3}, z=\frac{1}{27}$ ) corresponds to the point $A$ in the ( $a, b, c$ ) plane ( $a=b=c=\frac{1}{3}$ ).
The tangent at the cusp is $y^{\prime}=3 z^{\prime}$. The curve cuts the $y$ axis at $F\left(y=\frac{1}{3}, z=0\right)\left(a=0, b=c=\frac{1}{2}\right)$ and has ordinary contact with it at the origin corresponding to the vertex of the reference triangle in the fundamental region in ( $a, b, c$ ). We locate also the points

$$
\begin{aligned}
& Z ; a: b: c:: 2: \mathrm{I}: \mathrm{I}, \quad y=\frac{5}{16}, \quad z=\frac{1}{32} \\
& H ; a: b: c::-\mathrm{I}: \mathrm{I}: \mathrm{I}, \quad y=-\mathrm{I}, \quad z=-\mathrm{I} \\
& B ; a: b: c:: \frac{(3-1 / 17)}{4}: \frac{(\mathrm{x}+1 \mathrm{I})}{8}: \frac{(\mathrm{I}+1 \mathrm{I} 7)}{8}, y=\frac{(5117-19)}{3^{2}}, \quad z=\frac{(5-3 \sqrt{17})}{64} \\
& C \text { the conjugate of } B .
\end{aligned}
$$

$B$ and $C$ occur in the solution of the equilateral triangle case and are on both $D_{1}=0$ and $D_{2}=0$.
The locus $D_{2}=0$ becomes in the $(y, z)$ plane

$$
\begin{equation*}
(y-2 z)^{2}(y+2 z)-4 z(y-2 z)+z=0 . \tag{45}
\end{equation*}
$$

This cubic has asymptotes

$$
\begin{array}{ll}
y-2 z-\frac{1}{2}=0 & \text { with a cusp at infinity, } \\
y+2 z+1=0 & \text { with the third intersection at } z=-\frac{1}{3} .
\end{array}
$$

At the origin the inflexion $y^{3}+z=0$ gives a three-point contact with $D_{1}=0 . \quad D_{1}$ and $D_{2}$ also touch at $B$ and $C$. They cut at $H$ and after entering at $H$ the real region ( $D_{1}>0$ ), $D_{2}$ remains continuously within $D_{1}$ to the infinite line. The part of $D_{2}$ without $D_{1}$ is of course not represented in the real $(a, b, c)$ plane.

The locus $T=0$ in the $(y, z)$ plane is

$$
\begin{equation*}
4 y^{2}-20 y z+40 z^{2}-y+z=0 \tag{39}
\end{equation*}
$$

This ellipse meets $D_{\mathrm{I}}$ and $D_{2}$ at the origin and meets $D_{\mathrm{I}}$ at $F$. It lies partly within and partly without the real region $\left(D_{1}>0\right)$. The cusp of $D_{\mathrm{r}}$ the point $A$ is within the ellipse. $T=0$ also touches $y-z=0$ at the origin.

The locus $\pi(a+b)=0$ becomes $y-z=0$
The locus $\pi(a+b-c)=0$ becomes $4 y-8 z-\mathrm{I}=0$ and touches $D_{\mathrm{I}}$ at $F$.
The locus $a b c=0$ becomes $z=0$.
Since the transformation of the fundamental region is point for point there is no trouble in transferring the compartment markings from the ( $a, b, c$ ) plane. For new compartments we have, introducing as dividing curves,


$$
\begin{align*}
& H_{\text {I }} .4 y^{2}-8 y z+z-y=0 \quad(a=0, \beta=0)  \tag{49}\\
& C_{1} \cdot y^{13}-2 y^{2} z+3 y z-5 z^{2}-z=0 \quad(\alpha=\infty)  \tag{50}\\
& \text { • } W_{\mathrm{I}}: D_{\mathrm{I}}<0, y-z<0, \dot{H}_{\mathrm{I}}<0 \\
& M_{2}: D_{\mathrm{I}}<0, y-z<0, H_{\mathrm{I}}>0 \\
& X X_{I}: D_{\mathrm{I}}<0, y-z>0, T<0, y-2 z<0 \\
& X X_{2}: D_{1}<0, y-z>0, T>0, y-2 z<0 \\
& X X X^{\prime}: D_{1}<0, y-2 z>0 T<0,4 y-8 z-1<0 \\
& X X X: D_{\mathrm{I}}<0, y-2 z>0, T>0,4 y-8 z-\mathrm{I}<0 \\
& X_{1}: D_{1}<0,4 y-8 z-\mathrm{I}>0, T>0 \\
& X_{2}: D_{1}<0,4 y-8 z-1>0, T<0, z>0 \\
& P_{\mathrm{I}}: D_{\mathrm{I}}<0, z<0, D_{2}>0, C_{1}>0, H_{\mathrm{I}}<0 \\
& P_{2}: D_{1}<0, z<0, D_{3}>0, C_{1}>0, H_{1}<0 \\
& P_{3}: D_{2}<0, z<0, D_{2}<0, C_{1}>0, H_{1}<0 \\
& M_{\mathrm{I}}: D_{\mathrm{I}}<0, z<0, D_{2}<0, C_{\mathrm{I}}>0, H_{\mathrm{I}}<0 \\
& P_{\mathrm{r}}^{\prime}: D_{\mathrm{r}}<0, z<0, D_{2}>0, C_{\mathrm{I}}<0, H_{\mathrm{r}}>0 \\
& P_{3}{ }^{\prime}: D_{\mathrm{r}}<0, z<0, D_{2}<0, C_{\mathrm{r}}<0, H_{\mathrm{r}}>0 \\
& W_{\mathrm{r}}^{\prime}: D_{\mathrm{r}}<0, z<0, D_{2}<0, C_{\mathrm{r}}>0, H_{\mathrm{r}}>0
\end{align*}
$$

In tracing the intersections of the curves $C_{\mathrm{I}}$ and $H_{\mathrm{I}}$ with themselves and the other curves of the $(y, z)$ plane we notice that $C_{r}$ passes the origin with an inflexion $y^{3}-z=0$ and so goes from $\oplus$ to 000 . Since the asymptote $4 y-8 z+1=0$ lies entirely in 000 in the third quadrant and the curve crosses it at $z=-\frac{1}{96}$ and does not cross the $y$ axis, it must remain in this compartment to infinity. Leaving the origin in the first quadrant the curve does not cross $F Z$ $(4 y-8 z-\mathrm{I}=0)$ but crossing $D_{\mathrm{s}}$ enters $X X X$. As it reaches $y=2 z$ at $J(2, \mathrm{I})$ which is outside $T$ it must next cross $T$, pass through $X X X^{\prime}$, and enter $X X_{I}$ at $J$ and remain in this region to infinity. This serpentine branch of $C_{\mathrm{Y}}$ cuts $H_{\mathrm{I}}$ at the origin and in two other real points, one in $X X X$ and one in $X X X^{\prime}$. The other branch is parabolic, approximating $y^{2}+5 z=0$, and lies in the third and fourth quadrants. It crosses the $z$ axis at $z=-\frac{1}{6}$, outside $D_{\mathrm{r}}$ at whose crossing $z=-\frac{4}{z} \frac{1}{\eta}$, and meets $D_{1}$ at $I I y=-1, z=-1$ where the curves have ordinary contact. There is no further intersection in these quadrants.

The existence of the compartments named is thus proved.
In transforming to the $(\phi, \tau)$ plane we have the birational transformation

$$
y=-\frac{\phi(\phi-\tau+1)}{4 \tau(\phi-\tau-1)}, \quad z=-\frac{\phi(\phi-\tau)}{4 \tau(\phi-\tau-1)}
$$

with the reverse

$$
\begin{equation*}
\phi=\frac{4 z(y-2 z)}{(y-z)(4 y-8 z-1)}, \quad \tau=\frac{z}{(y-z)(4 y-8 z-1)} \tag{9}
\end{equation*}
$$

Real $(y, z)$ 's give real $(\phi, \tau)$ 's and conversely.
Complex $(y, z$ )'s give real ( $\phi, \tau$ )'s only for $y=2 z ; \phi=0, \tau=-1$. The whole linear locus $y-2 z=0$ complex as well as real is thus represented at the point $\phi=0, \tau=-1$, but no other complex points in the $(y, z)$ plane give real points in $(\phi, \tau)$. Other singular lines of the transformation are
$z=0$ which becomes the point $\phi=0, \tau=0$ but as Limit $\frac{\phi}{\tau}=y$ the linear elements at the point represent the various values of $y . \quad y-z=0$ gives $\phi$ and $\tau$ infinite with Limit $\frac{\phi}{\tau}=-4 z$. $4 y-8 z-\mathrm{r}=\mathrm{o}$ also gives $\phi$ and $\tau$ infinite but Limit $\frac{\phi}{\tau}=\mathrm{I}$ unless $y$ and $z$ are infinite when $\phi-\tau=\mathrm{r}$. As before noted this line $\phi-\tau-\mathrm{I}=0$ occupies a unique position inasmuch as to every point on it corresponds the same triangle, namely the limit of

$$
a: b: c:: \frac{1}{2}: p:-p+\frac{1}{2}
$$

when $p$ is infinite.
The line $y-2 z=0$ gives $\phi=0, \tau=-1$ and has in the limit $\frac{\phi}{\tau+1}=\frac{4 z}{4 z-\mathrm{r}}$, assigning real linear elements to real $(y, z)$ 's, complex to complex.

The boundary curves in the ( $\phi, \tau$ ) plane are (Fig. 7):

$$
\begin{equation*}
D_{\mathrm{L}}=(\phi-\tau)^{2}(\phi-4 \tau)^{2}+(\phi-\tau)\left(3 \phi^{2}+28 \phi \tau-32 \tau^{2}\right)+(\phi-\tau)(3 \phi-16 \tau)+\phi=0 \tag{52}
\end{equation*}
$$

The asymptotes are

- $\quad \phi-\tau=0$, intersecting also at ( 0,0 )

$$
\phi-\tau=\frac{1}{9}, \text { intersecting also at }\left(\frac{1}{2} 7^{2}, \frac{12}{2} 7^{5}\right)
$$

Corresponding to the factors $(\phi-4 \tau)^{2}$ is a parabolic asymptote

$$
3(\phi-4 \tau)^{2}+128 \tau=0 .
$$

At $\phi=-\mathrm{r}, \tau=\mathrm{o}$ is an inflexion $\phi^{\prime 3}=54 \tau^{\prime}$ which has four-point contact with the curve $\beta=\frac{(\phi+\mathrm{r})^{3}}{\tau}$ $=54$ at this point. At $\phi=0, \tau=-1$ is a conjugate point. The point $A$ is represented by a cusp at $\phi=\frac{7}{2}, \tau=\frac{27}{8}$. At $B\left(\phi=-\frac{(13+31 \overline{17})}{4}, \tau=-\frac{(45+11 \gamma \overline{17})}{16}\right)$ the curve has contact with $D_{2}$ and also at $C$ the conjugate of $B$ in ( $a b c$ ). The point $H$ becomes the infinite point on $\phi-4^{\tau}=0$, the axis of the parabolic branch.

The locus $D_{2}$ is

$$
\begin{equation*}
(\phi-\tau)(\phi-4 \tau)(3 \phi-4 \tau)+\phi^{2}=0 \tag{53}
\end{equation*}
$$

Its asymptotes are

$$
\begin{array}{rll}
\phi-\tau+\frac{1}{3} & =0 \text { with intersection at } \phi=-\frac{1}{3}, & \tau=0 \\
\phi-4 \tau+\frac{2}{3}=0 \text { with intersection at } \phi=\frac{2}{25}, & \tau=\frac{1}{5} \\
3 \phi-4 \tau-2=0 \text { with intersection at } \phi=2, & \tau=1
\end{array}
$$

At the origin is a cusp $\phi^{2}=16 \tau^{3}$.
The contacts of the curve with $D_{1}$ have been noted. At $\phi=2, \tau=1$, which is an infinite point in $(a, b, c)$ and $(y, z), D_{2}$ touches $\phi-\tau-1=0$, the line which also falls on $D(\alpha, \beta)$ in the $(\alpha, \beta)$ plane.

The locus $T=0$ is a hyperbola

$$
\begin{equation*}
\tau(6 \phi-1)-\left(6 \phi^{2}-3 \phi+1\right)=0 \tag{54}
\end{equation*}
$$

the asymptotes being

$$
\phi=\frac{1}{6}, \quad \phi-\tau=\frac{1}{3} .
$$




The complete representatives of $D_{\mathrm{s}}$ and $D_{2}$ are the irreducible factors above set out with the addition of $\phi=0$ in both cases and also $\phi-\tau=0$ in the case of $D_{1}$. These factors which vanish with $\alpha$ may be properly set aside and treated with the zero and infinite values of $\alpha$ and $\beta$ as nonfinite discriminantal factors.

The ( $\phi, \tau$ ) representatives of $z=0, y-z=0,4 y-8 z-1=0$ have been discussed ( p .27 ), the curve $H_{\mathrm{I}}$ becomes $\phi+1=0$. The curve $C_{1}(a=\infty)$ becomes

$$
\begin{gather*}
16 \tau^{4}+\tau^{3}(-40 \phi+16)+\tau^{2}\left(33 \phi^{2}-28 \phi\right)+\tau\left(-10 \phi^{3}+10 \phi^{2}\right)+\phi^{2}(\phi+1)^{2} \equiv \\
(\phi-\tau)^{2}(\phi-4 \tau)^{2}+2(\phi-\tau)\left(\phi^{2}+6 \phi \tau-8 \tau^{2}\right)+\phi^{2}=0 \tag{55}
\end{gather*}
$$

The asymptotes parallel to $\phi-\tau=0$ are complex and the infinite point on this line is a conjugate point. The infinite branch corresponding to $\left(\phi-4^{\tau}\right)^{2}$ is parabolic. As in $(y, z)$ the curve has two separate branches, and therefore at least a square root must be used in expressing the points. This is sufficient, for writing

$$
\phi-\tau=x, \quad \phi-4^{\tau=y}
$$

we have

$$
\frac{y}{2 x}=\frac{11 x+2 \pm 1-144 x^{3}+117 x^{2}+36 x}{9 x^{2}-2 x+1}
$$

The quadratic denominator is essentially positive and so $y$ is infinite only for infinite $x$ 's, that is on the parabolic branch.

From the radical, limits for the branches are obtained.
The closed branch has a cusp at the origin, $16 \tau^{3}+\phi^{2}=0$.
The open branch touches $\tau=0$ and so $D_{\mathrm{I}}$ at $P(-\mathrm{I}, 0)$.
The closed branch has three-point contact at the origin with $D_{1}$, and $D_{1}$ is closer to the $\tau$ axis than $C_{r}$ for points in the third quadrant near the origin. The remaining intersections with $D_{1}$ are the infinite points on $\phi-\tau=0$, and $\phi-4^{\tau=0}$; a two-point contact at $J(0,-1)$ which is a conjugate point on $D_{\mathrm{r}}$, and a single intersection approximately at $\phi=-.27$, $\tau=-.15$, which is better determined by the rational value for $\beta,-\frac{2}{4}$. There are also two complex intersections.

The curve $C_{1}(\alpha=\infty)$ meets $D_{2}=0$ at the origin, at infinity on $\phi-\tau=0$, at infinity on $\phi-4^{\tau}=0$, at a point where $\beta=-4^{7} \quad(\phi=-.4653 \cdots, \tau=-.0226)$ and two complex points which with this are the roots of an irreducible cubic. At these three points the curves have an ordinary contact.

The locus $\beta=\frac{(\phi+1)^{3}}{\tau}=-2_{4}^{2}$ passes successively through the cuts of $a=\infty$ and $D_{1}, D_{2}$, $\phi-\tau-I=0$. The latter point is $\left(+\frac{1}{2},-\frac{1}{2}\right)$.

The cuts of $\alpha=\infty$ with $\phi+1=0, \phi=0$ loci for which $\alpha=0$ give as points where $a$ must be determined by the direction of approach,

$$
I:(0,0), \quad J:(0,-1) \text { for } \phi=0
$$

and the points

$$
\tau=0,-.609 . .,-1.245 .,-1.646 \ldots \text {. for } \phi=-1 .
$$

To complete the essential features of the diagram we notice that $D_{1}$ has a closer contact with the $\tau$ axis at the origin than $D_{2}$.

The line $\phi=-1$ meets $D_{\mathrm{r}}$ at $\tau=-1.014$, meets $T$ at $\tau=-\frac{1}{7}$, and meets $D_{2}$ at $\tau=$ -. I33. .

These points enable the ordering of certain compartments to be made clear.
The $(\phi, \tau)$ plane is covered as by a single sheet of the $(y, z)$ plane stretched but not folded. Continuity is preserved except for the singular points and lines of the transformation, which become lines and points respectively, and except that as regards the infinite values the usual conventions of the projective plane are to be observed. It is convenient to locate some special points:

$$
\begin{aligned}
& A: \phi=\frac{\tau}{2}, \tau=\frac{2 \tau}{8} \\
& B: \phi=-\frac{(13+3 \sqrt{17})}{4}, \tau=-\frac{(45+11 \sqrt{17})}{16} \\
& C: \text { conjugate to } B \\
& D: \phi=2, \quad \tau=1 \\
& E: 3 \phi=4 \tau=\infty \\
& F: \phi=\tau=\infty \text { on } \phi-\tau=0 \quad \text { i.e., on } D_{1} \\
& G: \phi=\tau=\infty \text { on } \phi=\tau+\frac{1}{3}=0 \text { i.e., on } D_{2} \\
& H: \phi=4 \tau=\infty \\
& I: \phi=\tau=0 \\
& J: \phi=0, \tau=-1 \\
& L: \phi=3 \sqrt{5}-\mathrm{I}, \quad \tau=\frac{5}{2} \sqrt{5} \text { and } M \text { its conjugate } \\
& N: \phi=5, \quad \tau=4: L, M, N \text { occur in the case of } a=4, \quad \beta=54 \\
& P: \phi=-1, \quad \tau=0 \quad \alpha=4, \beta=54 \\
& W: \phi=2.629 \ldots \quad \tau=.886 . . \quad a=4, \quad \beta=54 \\
& Z: \phi-\tau=\frac{1}{9}, \quad \phi=\infty \text { on an asymptote of } D_{1} .
\end{aligned}
$$

The region (I) is within the cusp of $D_{\mathrm{I}}$ at $A$ and reaches to the infinite with $F$ and $Z$ as limits of the branches.

The region (4) is identified by means of $H, D$, and $W$, which are on its boundary, while $E$ is not.

The region (3) is bounded entirely by $D_{2}$ and infinite points and reaches $D, H$, and $E$ and is then inside the branch of $D_{2}$ in the first quadrant of $(\phi, \tau)$.

The region (2) is located by $E$ and $N$ and by its separation from (3) by $D_{2}$.
The regions (5), (6), (7) all reach $C$ and $(5)$ is bounded entirely by $D_{\mathrm{I}}$ and $D_{2} ;(6)$ reaches $I$ with ( 5 ), while ( 7 ) does not. ( $7^{\prime}$ ) which in $(y, z)$ has continuity with (7) through infinity, has in $(\phi, \tau)$ continuity through the point $P$ and joins (4) in the infinite regions in ( $\phi, \tau$ ) just as it joins (4) in ( $y, z$ ) along $y-z=0$.

The regions (8), (9), (10) have contact of their boundaries at $B$, and (9), being entirely bounded by $D_{2}$, is the inside region in $(\phi, \tau)$. (8) reaches $H$ in $(y, z)$ and so must belong to the parabolic branch of $D_{2}$ and be the upper one of the three regions.

Of the regions with real sides and imaginary bisectors $\oplus$ which in $(y, z)$ reaches $F$ and $Z$ and is bounded by $D_{1}$ and $z=0$ in ( $\phi, \tau$ ) is bounded by $D_{\mathrm{r}}$ and $\phi-\tau=0$ and reaches $I$.

- bounded in $(y, z)$ by $D_{2}, z=0$ and $4 y-8 z-\mathrm{I}=0$ in $(\phi, \tau)$ lies in the first quadrant between $D_{2}$ and $\phi-\tau=0 . \quad 00$ is continuous with 0 through $D_{2} . \quad 000$ in $(y, z)$ joins 00 along $y-z=0$ which involves $\phi=\infty, \tau=\infty$ with $\frac{\phi}{\tau}=-4 z$ and $z$ runs from zero to $-\frac{1}{4}$ so in $(\phi, \tau)$ these regions
are continuous through infinity in the second and sixth octants. ooo reaches $I$ by passing through $J$ a singular point. In $(y, z) 000$ is continuous with $X X X^{\prime}$ through infinity, in $(\phi, \tau)$ through the line $\phi-\tau-I=0$.

The regions outside $D_{1}$, that is corresponding to complex sides, are to be identified as follows:
The line $H_{\mathrm{z}}=0$ becomes $\phi+\mathrm{I}=0$ and the origin in $(y, z)$ has linear elements which cover $\phi=0, \tau>0$ hence $M_{\mathrm{a}}$ bounded also by $D_{\mathrm{r}}$ and reaching $C$ is identified.

For the region $W_{I}$, the asymptote of $H_{\mathrm{I}} y=\frac{1}{8}$ reaches infinity at a point which becomes $P$ and as $W_{I}$ reaches $y=z$ for all positive values of $z$ in the $(\phi, \tau)$ plane it must reach all infinite $(\phi, \tau)$ 's which are the limits of $\phi=-4 z \tau$ and so occupies the remainder of the second quadrant after $M_{2}$ is removed. The same argument locates $X X_{1}$ and so $X X_{2}$ by crossing $T$ but not $\phi=0$. 'The regions $X X X$ and $X X X^{\prime}$ can now be reached through $J$, the linear boundaries in $(y, z)$ being replaced by the collection of linear elements at $J$, which is a singular point of the transformation.

In the $(\phi, \tau)$ plane it is convenient to subdivide these compartments by means of the lines $H_{1}$ and $C_{1}$ representing zero and infinite values of $\alpha$.
$X_{1}$ is identified by $A$, and the cusp on $D_{I}$ and $X_{2}$ by crossing $T$.
$\oplus$ is reached from $X X X$ by crossing $D_{\mathrm{r}}$.
$M_{I}$ is reached from $7^{\prime}$ by crossing $D_{I}$, thence crossing $C_{I}(a=\infty)$ we arrive at $P_{3}$; across $H_{\mathrm{I}}(a=0)$ to $P_{3}^{\prime}$, across $\alpha=\infty$ to $W_{\mathrm{I}}^{\prime}$, across $D_{2}$ to $W_{2}$, across $\alpha=\infty$ to $P_{\mathrm{I}}{ }^{\prime}$, across $a=0$ to $P_{\mathrm{I}}$, across $\alpha=\infty$ to $P_{2}$ (see Fig. 8).

It is convenient to subdivide also the regions $\oplus$ by $\phi=-\mathrm{I}, \infty$ by $T, \infty 00$ at $J$ and again by $a=\infty$.

In the $(a, \beta)$ plane the discriminantal loci to be traced are:

$$
\begin{aligned}
& D(\alpha, \beta) \text { the representative of } D_{\mathrm{I}}, D_{2} \text { and the line } \phi-\tau-\mathrm{I}=0 \\
& T(\alpha, \beta)=0 \text {, the axes and the infinite boundary (Fig. 9). }
\end{aligned}
$$

On account of the single value of $\alpha$ for given $\beta, \sigma$ and the connection with a model of the surface $F(\sigma, \alpha, \beta)=0$ the $\alpha$ axis is taken vertical. To trace $T=0$ which is a rational curve in $(\phi, \tau)$ the parametric representation, obtained by substituting for $\tau$ its rational expression in $\phi$, namely $\frac{\left(6 \phi^{2}-3 \phi+1\right)}{(6 \phi-1)}$ in the ( $\left.\phi, \tau\right)$ expressions for $\alpha, \beta$ (10).

This gives

$$
\left.\begin{array}{l}
a=\frac{(\phi+1)^{2}(6 \phi-1)^{3}(2 \phi-1)}{43^{2} \phi^{5}-428 \phi^{3}+312 \phi^{2}-102 \phi+15} \\
\beta=\frac{(\phi+1)^{3}(6 \phi-1)}{6 \phi^{2}-3 \phi+1}
\end{array}\right\}(57)
$$

Letting $\phi$ range over all real numbers we have a real branch. It is proper however to inquire whether any other branch exists by which complex ( $\phi, \tau$ )'s are represented by real $(\alpha, \beta)$ 's.

The general condition that such may be the case is :

$$
\begin{align*}
& \text { If } a=f(\phi)=f(a+b i)=A+B i  \tag{58}\\
& \text { and } \beta=g(\phi)=g(a+b i)=C+D i \text {, }
\end{align*}
$$

$B$ and $D$ have a common factor other than $b$.

In this case $\beta$ is real if $b=0$ or if

$$
\begin{align*}
& 3(4 a-\mathrm{r})\left[6 b^{4}-3 b^{2}\left(\mathrm{I} 2 a^{2}+17 a+5\right)+(6 a-\mathrm{r})(a+\mathrm{r})^{3}\right] \\
& +\left[6 b^{2}-\left(6 a^{2}-3 a+\mathrm{r}\right)\right]\left[-b^{2}(24 a+\mathrm{r} 7)+3(8 a+\mathrm{I})(a+1)^{2}\right]=0 \tag{59}
\end{align*}
$$

The coefficient of $b^{4}$ is $(-72 a-120)$ and a set of terms free from $b$ exists.
For $a=1$ the expression becomes

$$
-192 b^{4}-106 b^{2}+72
$$

which does not reduce, and hence the general expression does not reduce.


For $a=0, b^{2}=\frac{2}{3}, \beta$ is real but the corresponding expression for $\alpha$ is

$$
\left[432 b^{6}+288 b^{4}+105 b^{2}-\mathrm{r}\right]\left[432 b^{4}+428 b^{2}-102\right]-\left[432 b^{4}+164 b^{2}-18\right]\left[312 b^{2}-15\right]
$$

when $a=0$, and this is not zero when $b^{2}=\frac{2}{3}$.
Hence there is no such common factor and $T(\alpha, \beta)=0$ is a unicursal curve and $\phi$ is a proper parameter.

It is moreover in I : I correspondence with the hyperbola in the $(\phi, \tau)$ plane, and the facts discovered there may be utilized in tracing.

The curve $T$ meets $\alpha=\infty$ in one point only, for at $J$, which is apparently an intersection, the linear elements differ and $J$ is singular.

This gives a single asymptote $\beta=.0164$. . . approximately.
$\beta$ is infinite for no real finite $\phi$ but for $\phi=\infty, \alpha=\infty, \beta=\infty$.
This gives a parabolic branch with $\beta=\alpha^{2}$ as parabolic asymptote.


For $\phi=-1, a=0, \beta=0$ the character being given by

$$
5^{4} \cdot 17^{3} \cdot a^{3}=2^{2} \cdot 3^{3} \cdot 7^{7} \cdot \beta^{2} \quad \text { a cusp. }
$$

At $\phi=\frac{1}{6}, a=0, \beta=0$

$$
2^{4} \cdot 3^{6} \cdot \beta^{3}+7^{7} \cdot \alpha=0 \quad \text { an inflexion. }
$$

Since the curve crosses $D_{2}$ at the real four-point it must touch $D(\alpha, \beta)$. This occurs at

$$
\alpha=-.99678 . \quad \beta=5.3542 . \quad \text { approximately. }
$$

For $\phi=\frac{1}{2} ; \alpha=0, \beta=\frac{2}{4}$.
For $\phi=0 ; \alpha=1_{15}^{1}, \beta=-1$.
For $\beta=-1 \phi^{2}=0$ or $\phi$ is complex. This point which corresponds to $J$ is then a turningpoint for $\beta$. For $\beta<-1$ all the values of $\phi$ are complex.

There is a maximum value for $a$ at $\phi=-.33$. ., $a=.66 . ., \beta=-.33$, and a maximum value at $\phi=.402$. ; $\alpha=-1.01$. , $\beta=5.45$., and a maximum value at $\phi=-1.58$. $; \quad \alpha=-.8307 ., \beta=.3012$. . The curve meets $D$ only at the origin and the four-point.

These and the negative facts implied by omission are sufficient to establish the general character of the curve.

In the $(\phi, \tau)$ plane the curves $(\phi+1)^{3}-\beta \tau=0$ form a family of cubic parabolas with centers at $P(-1,0)$ and can be easily visualized in the diagram. By so doing it is seen that the numbering of the regions $x, 2,3,4,5,6,7,8,9,10$ corresponds to the order of magnitude of the real roots of $F(\sigma, a, \beta)=0$.

Since $D_{\mathrm{I}}$ has three-point contact with the curve $\beta=54$ of this family at $P$, and $\alpha$ here indeterminate has the limit 4 for this approach, the region $7^{\prime}$ is seen to contain only values of $\beta$ which are greater than 54 and to be continuously joined to (7) along $a=4$ in the ( $a, \beta$ ) plane. This corresponds to the change of class of the root from (113) to (331) when a passes the value 4 (§8).

Digressing to complete the comparison of the two classifications of the real roots we have the set ( 122 ) $[\$ 8]$ has one negative side and as the sum of the positive sides is not greater than 2, the negative side cannot be less than -1 . This identifies the set of three with (8), (9), (10). (122) has the greatest bisector internal and opposite $a_{\mathrm{r}}$ : hence $\left|a_{1}\right|$ is the least magnitude among the sides and as it approaches zero the other sides approach $\frac{1}{2}, \frac{1}{2}$ which is the triangle represented at F . This identifies ( 122 ) and ( I ). ( 22 I ) can approach $H(1,1,-1$ ) and is then (8): (212) is (9). The set (23I), (3I2), (32I) correspond to (2), (3); (4), and (2) reaches the line $a+b-c=0$ or has a side equal to $\frac{1}{2}$. This must be $c_{2}$ for if $a_{2}$ or $b_{2}$ has this value $c_{5}$ is infinite. Thus (2) is (312). (4) reaches $a+b=0$ or one side has the value 1 . This must be $a_{2}$. So (4) is (23I) and (3) is (32I).

For the set (5), (6), (7), consider the approach to ( $1,0,0$ ): only $a_{x}+b_{8}+c_{3}$ can reach this point with $a$ and $b$ negative and nearly equal : (II3) is (5). (13I) can approach with $a$ and $c$ negative and very unequal, this is then (6). (3II) is then (7), while ( $7^{\prime}$ ) having two positive sides greater than $I$ and a negative side less than $-I$ is ( 33 r ).

The transformation from the $(\phi, \tau)$ plane to the $(\alpha, \beta)$ plane effects a $10:$ r correspondence and brings as in general complex ( $\phi, \tau$ )'s into correspondence with real ( $\alpha, \beta$ )'s.

Beginning with the real regions of the ( $\phi, \tau$ ) plane it is necessary to determine the limiting values of $\alpha$ and $\beta$ at the points where they become indeterminate and also for the infinite values of $(\phi, \tau)$ by various paths of approach.

The limit of $\alpha$ for $\phi=m \tau=\infty$ is $\frac{4 m^{3}(m-1)}{(m-1)^{2}(m-4)^{2}}$.
The limit of $\alpha$ along $(\phi+1)^{3}-\beta \tau=0$ for finite $\beta$ as $\phi$ approaches $\infty$ is 0 .
The limit of $a$ along $\phi-m \tau^{2}=0$ is 4 .
The limit of $\alpha$ along $m \phi^{2}-\tau=0$ is 0 .
The limit of $a$ along $\phi(\phi-\tau)-\mathrm{I}=0$ is 4 .
For finite points:
At the origin-
The limit of $a$ along $\phi-m \tau=0$ is $\frac{4 m(m-1)}{m^{2}}$.
The limit of $\alpha$ along $D_{1}$ is 4: along $D_{2}$ is $\infty$.
At the point $P(-1,0)-$
The limit of $a$ for all rectilinear approaches is o.
The limit of $a$ for approach on any curve $(\phi+I)^{3}-\beta \tau=0$ is 4 .
The limit of $a$ for approach on $m(\phi+1)^{2}-\tau=0$ is $\frac{4}{(20 m+1)}$ and the value infinity occurs only for negative $\boldsymbol{\tau}$ 's.

At the point $J(0,-1)$ -
The limit of $a$ for approach on $\tau-n \phi+1=0$ is $\frac{1}{(3-4 n)}$.
The point $J$ then represents all real points on the line $\beta+\mathrm{I}=0$.
At the point $P(-1,0)-$
The limit of $\beta$ for approach not on a $\beta$ curve is $\infty$.
Any value of $\beta$ is reached by approach on the $\beta$ curve $(\phi+1)^{3}-\beta \tau=0$.
As a preliminary to identifying the compartments the signs of $\alpha, \beta$ may be marked on the ( $\phi, \tau$ ) diagram (Fig. 8).

The discriminantal loci in the $(\phi, \tau)$ plane are $D_{2}, T$ and the loci giving zero and infinite values to $\alpha$ and $\beta$.

In transferring the ( $\phi, \tau$ ) compartments to the $(\alpha, \beta)$ plane they must be folded at the proper discriminantal lines $D_{2}$ and $T$. (Though $D_{土}$ falls on $D[\alpha, \beta]$ it is not discriminantal.)

For the other loci a special inquiry must be made. For $\alpha=\infty$, e.g. the points on the same $\beta$ curve, close to and on opposite sides of this locus, correspond to $\alpha=+m, \alpha=-m$ and in the ( $a, \beta$ ) plane are near only in the projective sense: folding is then not the proper word.

If such a locus occurred with the vanishing or infinite factor entering with an even exponent, folding would suit, but the only one of this class for finite $(\phi, \tau)$ has a further characteristic which prevents the use of the concept. Namely, the locus $\phi+I=0$ contains only points for which $\alpha$ vanishes to the second order, while $\beta$ vanishes to the third. The whole locus is a singular line all of whose points find their representation at the origin in the $(\alpha, \beta)$ plane. It is only possible to say that the sheets are connected at this point.

For infinite ( $\phi, \tau$ ) an example can be found. The region $W_{1}$ is connected in the ( $\phi, \tau$ ) plane with $X X_{1}$ by passage through the infinite line along $(\phi+\mathrm{r})^{3}-\beta \boldsymbol{\tau}=0$ for any negative $\beta$. In both regions, however, $\alpha$ is positive and as the limit of $\alpha$ along the $\beta$ curve is o the order of the vanishing factor must be even and folding occurs.

In the same way for positive $\beta, \circ_{1}$ and $000_{1}$ are folded on $\alpha=0$, in the fourth quadrant of $(\alpha, \beta)$ since $a$ is negative in both compartments.

Starting with those regions of the ( $\phi, \tau$ ) plane which fall on the first quadrant of the $(a, \beta)$ plane, the region ( I ) inside the cusp of $D_{1}$ falls inside the cusp of $D$. Since $D_{1}$ is not discriminantal $X_{1}$ is continuous with (I) over the line $D$ up to the parabolic branch of $T$. Along this curve the points are readily identified by the parametric value $\phi$. Otherwise $X_{1}$ reaches $\phi-\tau=0$, that is $a=0$, and $X_{2}$, which is folded on $X_{\mathrm{r}}$ at the curve $T$, reaches $\tau=0$, that is $\alpha=4$, $\beta=\infty$, while $X_{1}$ along $\phi(\phi-\tau)-1=0$ reaches for infinite $\phi$ the same point (Figs. 9 and 7 and p. 35).

The line $\phi-\tau-\mathrm{I}=0$ falls as a whole on $D$ and the part in the first $\phi, \tau$ quadrant falls on the boundaries of the "cusp" region. The point $D(2, \mathrm{I})$ falls on the cusp and the lower part of the line reaching $\tau=0$ reaches $a=4, \beta=\infty$. The upper part reaches $a=\infty, \beta=\infty$.

The line $D_{2}$ also falls on $D$ and the hyperbolic loop in the first octant of $(\phi, \tau)$ falls on the boundaries of the "cusp" region.

The three regions (2), (3), (4) fold alternately and form a sort of pleat. (2) is continuous with $X_{2}$ along the upper branch in $(a, \beta)$. The upper branch of $D_{2}$ in $(\phi, \tau)$ falls on the lower branch of $D$ in $(\alpha, \beta)$ and vice versa. Next consider the region $P_{2}$ which falls on the first quadrant of $(\alpha, \beta) . \quad P_{2}$ reaches $\alpha=0, \beta=0$ along $\phi=-\mathrm{I}$ and reaches $\alpha=4, \beta=\infty$ along $\phi(\phi-\tau)-$ $\mathrm{I}=0$ for $\phi=\infty$, and reaches $\alpha=\infty, \beta=\infty$ along the parabolic branch of $D_{1}$. It reaches $\alpha=0, \beta$ any positive, along $\phi-\tau=0$ and $\beta=0, a$ any positive at the cut of $a=\infty, a=0, \phi=-\mathrm{I}$.
$P_{2}$ does not contain $T$ in $(\phi, \tau)$ and so in $(a, \beta)$ passes continuously over the branches of $T$ without change. The regions (8), (9), (10) join $P_{2}$ over $D_{1}$ in ( $\phi, \tau$ ) and are pleated in ( $\alpha, \beta$ ) over the cusp region. The branch of $D_{2}$ between (8) and ( 9 ) falls on the lower boundary of the cusp region, for it is continuous through infinity with the boundary of (2), (3). $B$ falls on the cusp. For the region $M_{1} a>_{4}$ and moreover $a$ has a greater value than belongs to the branch of $D_{\mathrm{I}}$ which separates $M_{\mathrm{I}}$ from ( $7^{\prime}$ ). The region $M_{2}$ reaches $a=4, \beta=54$ at $P$ by approach on the $\beta$ curve and so joins $M_{\mathrm{r}}$. Further $M_{2}$ reaches $\beta=0, a$ any value between 0 and 4 by approach at $P$ along parabolas (p. 35). $M_{5}$ reaches $\beta=0, a$ any value between 4 and $\infty$ in the same way. $M_{2}$ reaches $a=0, \beta$ any positive at the points along $\phi=0 . M_{2}$ embraces the regions (5), (6), (7) which are pleated on the cusp region, $C$ falling on the cusp, the fold of (5) and (6) falling on the upper boundary and the fold of (6) and (7) on the lower. (7) for which all points have $\boldsymbol{a}$ not greater than 4 is joined continuously to ( $7^{\prime}$ ) for which all points are not less than 4.

Since $M_{I}$ and $M_{2}$ do not contain $T$ in the $(\phi, \tau)$ plane they pass over it without change in the plane $(a, \beta)$. With the regions (5), (6), ( 7 ) , and $\left(7^{\prime}\right)$ which are pleated on the cusp region they form a continuous covering of the first quadrant of the $(\alpha, \beta)$ plane in the same manner that $P_{3}$ with the pleated regions (8), (9), (10) does.

The sheet $X_{2}$ with the regions (2), (3), (4) pleated behaves in the same fashion, while the shect $X_{\mathrm{r}}$ containing the region (I) giving the internal solution is without fold at the boundaries of the cusp region. There remains in the first quadrant the pair of regions $X X X_{\mathrm{r}}$ and $X X X_{\mathrm{r}}{ }^{\prime}$ between $a=\infty$ and the negative side of $\phi+\mathrm{I}=0$ and separated by $T$. These reach all positive
a's for $\beta=0$ at the indeterminate points for $a$ (p. 29) and folding on the asymptotic branch of $T$ cover the space between this line and the $\alpha$ axis.

The sheets $X_{\mathrm{r}}$ and $X_{\mathrm{a}}$ are folded at the parabolic branch of $T$ where $\phi>\frac{1}{2}$ and the complete account of the first quadrant of $(a, \beta)$ is: 10 real roots for the "cusp" region, 4 real roots between this and the parabolic branch of $T, 2$ real roots between the two branches of $T$, and 4 real roots between the asymptotic branch of $T$ and the $a$ axis.

It is interesting to notice the persistence of the root (I) for a region of the ( $a, \beta$ ) plane much more extensive than the region where it has an interpretation as a solution of the problem for real angle-bisectors.

Taking up the second quadrant of the ( $a, \beta$ ) plane the region $W_{\mathrm{r}}$ has $a$ not greater than 4 , $\beta$ any negative. $\quad W_{\mathrm{r}}^{\prime}$ is continuous with $W_{\mathrm{x}}$ through $P$ and as the $\beta$ curves with $P$ as origin are $\beta \tau=\sigma^{3}$ and the curve $\alpha=\infty$ is $20 \tau=\sigma^{2}$, all the $\beta$ curves for $\beta<0$ fall between $a=\infty$ and the $\phi$ axis. For $W_{1}{ }^{\prime}, a$ is not less than 4 . The region is bounded by $D_{2}$ which cuts $a=\infty$ at a point for which $\beta=-\frac{2}{4}$, and as this value is asymptotic for $D(a, \beta)=0$ and $D_{2}$ is discriminantal the curve $\beta=-\frac{27}{4}$ touches $\alpha=\infty$ at the point in question in the $(\phi, \tau)$ plane. For $\beta>-{ }_{4}^{2}$ the $\beta$ curves leave $W_{r}^{\prime}$ by crossing $a=\infty$. So $W_{\mathrm{s}}$ and $W_{\mathrm{r}}{ }^{\prime}$ together form a sheet covering the second quadrant of $(a, \beta)$ up to the line $D$. At this line the sheet is folded and returns from the fold as $W_{2}$. $W_{2}$ reaches $\alpha=\infty$ for $0<\beta<-\frac{2 \pi}{4}$, and reaches $a=0, \beta=0$ along $\phi+1=0$, and reaches $\beta=\infty, 0<a<4$ at $I(0,0) . W_{1}$ has these latter values in the infinite regions. $W_{1}^{\prime}$ and $W_{2}$ only join (6) and (7) at $a=4 \beta=\infty$.

The ( $\phi, \tau$ ) regions $X X X_{2}, X X X_{2}{ }^{\prime}$ are continuous with $X X X_{1}$ and $X X X_{1}{ }^{\prime}$ respectively through the cuts of $a=\infty$ and $\phi+\mathrm{i}=0$ where $\beta=0$ and $a$ depends on the path of approach. These paths are parabolas touching $\alpha=\infty$ since this is of the first order while $\phi+\mathrm{r}=0$ is of the second, and so in passing these points from one of the regions to the other a does not become infinite. The regions are also continuous with $X X_{2}$ and $X X_{1}$ respectively through $J . X X_{1}$ is continuous with $000_{2}^{\prime}$ over $\phi-\tau-\mathrm{I}=0$, the non-discriminantal representative of $D(\alpha, \beta)$. The whole set forms a double sheet covering the second quadrant of $(a, \beta)$ (with the exception of the part inside the loop of $T$ ) and the part of the first quadrant up to the asymptotic branch of $T$, as before mentioned. It is folded on $T$ from the cut of $T$ and $a=\infty$ [asymptotic point of $T$ in $(a, \beta)]$ through $\phi=-\mathrm{I}\left[\right.$ cusp of $T$ at $a=0, \beta=0$ ], to $J(0,-1)\left[a=1 \frac{1}{5}, \beta=-\mathrm{I}\right]$ and up to $\phi=\frac{1}{6}$, the asymptote of $T$ in ( $\phi, \tau$ ) [inflexion of $T$ at $\alpha=0, \beta=0$ ].

The sheets reach $a=0, \beta$ any negative as follows:
$0>\beta>-\mathrm{r}$ in $X X_{2}$ ( $J$ represents $\beta=-\mathrm{r}$ as a whole) and in $X X_{1}$ at infinite points on $\beta$ curves. They reach $\beta=0, \alpha$ any positive at the indeterminate points discussed above.

The region inside the hyperbolic branch of $D(a, \beta)$ is covered by $000_{2}^{\prime}$ in one shect and the part of $X X X_{2}$ between the cut of $D_{\mathrm{r}}$ and $a=\infty$ and the origin in the other.

The third quadrant in $(a, \beta)$ has as representatives in $(\phi, \tau)$ :
$\oplus_{2}$ and $X X X_{4}$ continuous over $D_{1}$.
$000_{2}$ and $X_{X X} X_{4}{ }^{\prime}$ continuous through $J$, and since $\phi-\tau-1=0$ is crossed at $J$ and also between $X X X_{4}{ }^{\prime}$ and $000_{5}$ there must be added that part of $000_{4}$ for which $\phi+1>0$. This part has $0>\beta>-1$ but joins $000_{2}$ for $\beta=-1$, a any negative at $J$.

As no discriminantal lines occur we have two separate sheets. $X X X_{4}$ is continuous ovèr $\beta=0$ with $X X X_{3}$, and $X X X_{3}{ }^{\prime}$ with $X X X_{4}{ }^{\prime}$ over the same line.
$P_{1}^{\prime}$ and $P_{3}^{\prime}$ are folded on $D$ and reach $\beta=0, \alpha$ any negative at $P$ and the indeterminate point for $a$. At the fold $0>\beta>-\frac{27}{4}$ (Fig. 7). These facts locate the regions in $(a, \beta)$.

The fourth quadrant has in $(\phi, \tau)$ :
The second octant of ( $\phi, \tau$ ) having four regions $\circ_{I}, \mathrm{o}_{2}, \circ_{1}, \circ_{2}$ which meet at the four-point and are folded so as to hang together at the four-point in $(\alpha, \beta)$. They are joined in pairs along $D(a, \beta)$ for all values of $\beta$ greater than the four-point value, for lesser values the $\beta$ curves in $(\phi, \tau)$ meet $D_{2}$ in complex points.

For the lesser i's $\mathrm{OO}_{\mathrm{I}}$ and $\mathrm{OO}_{2}$ are folded on $T$. $\mathrm{O}_{\mathrm{I}}$ and $\mathrm{O}_{2}$ are folded on the other branch of $T$ up to ( $\phi=\tau=\frac{1}{2}$ ) [ $\left.\beta=\frac{2}{4} \frac{\tau}{2}, u=0\right]$ after which they pass continuously into $X_{I}$ and $X_{2}$ which are folded on this parabolic branch of $T$ in the first quadrant of $(\alpha, \beta)$.

The fold of $\mathrm{OO}_{I}$ and $\mathrm{oO}_{2}$ passes through the origin in $(\alpha, \beta)$ and is continuous with the fold of the sheets away from the loop of $T$ and reaches the first quadrant as the fold along the asymptotic branch of $T$.

The regions $P_{1}$ and $P_{3}$ are folded on $D_{2}$ for positive $\beta$ 's and reach $\alpha=\infty, \beta=\infty$ with $D_{2}$ and $\alpha=\infty, \beta$ any positive on the curve $\alpha=\infty$. They then cover all the fourth quadrant below the branch of $D$. They are continuous with $P_{1}^{\prime}$ and $P_{3}^{\prime}$ over the $\alpha$ axis.

The regions $X X X_{3}$ and $X X X_{3}{ }^{\prime}$ are folded on $T$, reach $\alpha=\infty, \beta=\infty$ along $\phi-\tau=0: \alpha=\infty$, $\beta=0$ at $\phi+\mathrm{I}=0$ but for $\alpha=\infty$ the greatest $\beta$ is .OI64 . . the value for the asymptote of $T$.

In $(\alpha, \beta)$ then these sheets are folded on the lower branch of $T$ and extending upward pass continuously into $\oplus$ and $000_{\mathrm{I}}$ respectively across $D_{\mathrm{I}}$ and $\phi-\tau-\mathrm{I}=0$, both of which fall nondiscriminantally on $D(\alpha, \beta)$.

Collecting the parts of the plane which are continuously connected in ( $\phi, \tau$ ) as well as in $(\alpha, \beta)$ we have, setting aside the region of ten real roots:

Sheet A containing $X_{I}, O_{I}$
Sheet B containing $X_{2}, \mathrm{o}_{3}$
Sheet C containing $M_{1}, M_{2}, \circ_{2}, W_{1}, W_{1}^{\prime}$
Sheet D containing $P_{2}, \oplus_{1}, \oplus_{2}, W_{2}, X X X_{3}, X X X_{4}$
Sheet E containing $\mathrm{ooO}_{\mathrm{I}}, \mathrm{ooo}_{2}, X X_{2}, X X X_{1}, X X X_{2}, X X X_{3}{ }^{\prime}, X X X_{4}{ }^{\prime}$
Sheet F containing $000^{\prime}{ }^{\prime}, X X_{\mathrm{I}}, X X X_{\mathrm{I}}{ }^{\prime}, X X X_{2}{ }^{\prime}$
Sheet G containing $P_{\mathrm{I}}, P_{\mathrm{P}_{1}}{ }^{\prime}$
Sheet H containing $P_{3}, P_{3}{ }^{\prime}$
Sheet J containing $\mathrm{OO}_{\text {: }}$
Sheet K, the tenth sheet, is not represented outside the region of ten real roots (Fig. 10).
The phenomena at infinity are:

$$
\text { I) } a=\infty, \beta \text { any finite value. }
$$

There are ten distinct finite roots except at the asymptotic lines of $T$ and $D$, any value of $\sigma$ being reached for at least two finite real values of $\beta$.

$$
\text { 2) } \beta=\infty, a>4 \text {. }
$$

Eight roots are infinite in pairs. For $\sigma^{2}=m \beta$ they are given by:

$$
16 a m^{4}-40 a m^{3}+33 \mathrm{am}^{2}+(4-10 a) m+(a-4)=0
$$

Two roots are $\sigma=1$, and these belong to the regions (5), (6) if $\beta$ approaches infinity from the positive side, and to the regions $X X X_{z}$ and $000_{2}{ }^{\prime}$ if $\beta$ approaches on the negative side.

The four pairs of infinite roots are (1) and (10):(2) and (9):(3) and (8): (4) and (7) for approach with positive $\beta$, while for negative $\beta$ all approach in the complex regions.

The equation in $m$ has no negative roots for $a>4$.
3) $\beta=\infty, \alpha=4$. One root is zero for any $\beta$. This is ( 7 ) or $\left(7^{\prime}\right)$.

Three roots are finite. These are easily determined from the $\phi, \tau$ equation for $a=4$ when $\beta$ is infinite and $\phi$ finite, for then $\tau=0$.

The equation is

$$
\tau\left[(\phi-\tau)(3 \phi-4 \tau)^{2}+4(\phi-\tau)(3 \phi-4 \tau)-\phi\right]=0 .
$$

Neglecting the indeterminate solution $\tau=0$, there are three values for $\phi: 0, \frac{(2+1 / 5)}{3}, \frac{(2-1 / \overline{5})}{3}$.

## Fiq10.


$\phi=0$ belongs to (5), the negative value to (6) and the positive to (4) for positive $\beta$. For negative $\beta$ they fall in $X X X_{2}, W_{2}, 000_{2}^{\prime}$ respectively.

The values obtained for $\sigma$ (or $\phi$ ), however, depend on the path of approach. If $\beta=\infty$ $a=4$ the roots are eight infinite and two indeterminate, while for approach along the lower branch of $D$ whose asymptote is $a-4=0$ the root (6) and the root (7) are continually equal and attain the limit $\phi=-\frac{1}{3}$.

For $3<a<4$ as $\beta$ approaches positive infinity there are four real roots, in the sheets $X_{1}$, $X_{2}, P_{2}, M_{2}$. Of these $X_{2}$ and $M_{2}$ have the limit $\sigma=\mathrm{I}$, the others being infinite. The three pairs of complex roots have an infinite limit.

For $a<0$ as $\beta$ approaches positive infinity six roots are real and two of these, $o_{2}$ and $\oplus_{2}$, are finite.

For $\beta$ approaching negative infinity the number and ordering of real and complex roots is essentially different, $\beta=\infty$ being a discriminantal line. For this approach and $\alpha>4$ two roots are real instead of ten. For $a<4$ but positive four roots are real and the finite pair are con-
tinuously connected $X_{2}$ to $W_{2}$ and $M_{2}$ to $X X X_{2}$. For negative a only two roots instead of six are real and these are the finite pair $\mathrm{OOO}_{2}$ (continuous with $\mathrm{OO}_{2}$ ) and $\oplus_{2}$ (continuous with $\mathrm{O}_{2}$ ). The discontinuity at $\alpha=4, \beta=\infty$ is essential, the original equation having its last three cocfficients indeterminate for these values.

The region of ten real roots terminates at the cusp where $(2)=(3)=(4),(5)=(6)=(7)$, $(8)=(9)=(10) . \quad$ A positive circuit of the cusp permutes the roots by the substitution (243) (567) (8, 10, 9).

On account of the essentially incomplete and non-analytic character of the real field a thoroughgoing application of the devices of a Riemann surface is of course impossible, nevertheless, as a means of presenting the complicated state of facts in a condensed form, it seems best to make a tentative use of them.

Drawing a barrier along $a=3$ from $\beta=27$ to $\beta=\infty$ and marking the above substitution on it, we name the ten sheets in correspondence with the ten real roots in the cusp region:

| A | B | C | D | E | F | G | H | J | K |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| I | 2 | 7 | 8 | 4 | 3 | 5 | 10 | 9 | 6 |

This is consistent with the barrier and gives the plan for the real roots shown in Fig. Io.
The complex roots need further inquiry. We conceive as many shects as needed laid over the regions in question and marked with the proper complex values. $D(a, \beta)=0$ is a locus of points such that three pairs of roots are equal. For the real region and also up to the four-point on the lower parabolic branch these have been determined. For the rest of the curve there are two equal pairs of complex roots. [For $k=l=\mathrm{r} D$ is rational in $m$ and the roots corresponding to $D_{2}$ are those of a squared cubic factor, whose discriminant vanishes at the four-points (one real) and for $m=0$ which corresponds to $\alpha=4, \beta=\infty$.] (See §17.)

It is proper so to name the complex roots that after a real pair become complex and of course conjugate the pairing should remain unchanged except by discriminantal points.

The part of $D_{2}$ where two complex pairs are equal will effeçt then an interchange of partners among the pairs, or rather the monodromie may be so ordered as to effect this.

In crossing $\beta=0$ ten roots become zero. In all cases, however, four are real and merely have their order of magnitude reversed. The three complex conjugate pairs have the order of the pairing inverted, the root with positive imaginary part and that with negative becoming interchanged in each pair.

In crossing $a=0$ six roots become infinite, and form a cycle. If we choose a pair to be real they must not be conjugate but opposite in the cycle. This effects a re-pairing discussed in the monodromie for $\beta=27$ ( $\$ 14$ ). Revising the numbers in respect of the barrier (3, 4) $(5,6)(9,10)$ leave the real axis as conjugates and for a path with $\beta$ constant crossing was proved impossible. Hence if $(0,4)$ be the real pair the hexagon is

and $(3,6)(5,10)$ are paired after crossing.

This is for positive $\beta$. A similar re-pairing occurs for negative $\beta$.
The phenomenon is unlike anything occurring on a Riemann surface for an analytic function of a complex variablc. There crossing a branch cut implies completion of a circuit round a discriminantal point: this passage is only a half-circuit in the a complex plane.

The cycles given for $a=\infty(\mathrm{I}, 2,5,8,4,3,9)(6,10,7)$ to be interpreted as cyclic substitutions invert the order of pairs but keep the pairing of complex roots.

For $\beta=\infty$ to connect the aproaches with the two signs the following system of interchanges is needed:

$$
\begin{aligned}
& 0<a<3(\mathrm{r})=(2),(3)=(4), \\
& (5,6)(9, \text { ro }) \text { change the order in the pairs, }
\end{aligned}
$$

(7) (8) change the relative order of magnitudc.

That is a half-circuit affecting pairs as on $\beta=0$.
$3<\alpha<4$ in addition to the above the barrier is to be extended with $(2,3,4)(7,6,5)$ ( $8,9,10$ ).
$a>4$ and to end of the upper branch of $D,(1)=(2),(6)=(5),(10)=(9),(7)=(8) . 。$
For $a=\infty, a^{2}>\frac{\beta}{4}$ up to the end of $T$ 's parabolic branch $(3)=(4),(7)=(8),(1)=(2)$, and also the half-circuit.

For the last two regions and also on the rest of the line $\beta=\infty, \alpha>0$ a barrier is needed with the substitution $(2,6)$ [or $(1,5)]$. For negative $a$ the order of magnitude is reversed and a barrier $(\mathrm{I}, 7)[$ or $(2,9)]$ is to be applied. This barrier and the $(2,6)$ barrier for $a>0$ is needed in view of the occurrence of that part of $D$ where two complex roots become equal.

With this set of conventions a consistent plan of the sheets can be drawn. In Fig. ir the discriminantal lines and the barriers are shown, with the equalities and conventional changes in brackets [ ] and the pairing of complex roots in parentheses (), the first of the pair being the root with positive imaginary part. The six-cycles at $a=0$ are symbolically indicated by hexagons.

Taken in connection with Fig. io for the real roots and the identification in p. 40, it is to be considered as a condensed expression of the various connections between the roots of the equation for the field of real $\alpha, \beta$.

Fig. io for the real roots may serve the purposes of a model of the surface $F(\sigma, \alpha, \beta)=0$ as far as the order of magnitude and number of the real roots for the various values of ( $\alpha, \beta$ ) is concerned.

There are, however, many questions which are proper to ask concerning the connections of the real roots which it does not answer or answers only with difficulty. For this reason a model is in order to complete the concise expression of the facts ( $\$ 19$ ).

The representation of the facts discovered in the field of complex $\alpha, \beta$ is of course out of the question. For instance there are two complex four-points, at which the loci $T=0$ and $D=0$ intersect. These loci are, however, continua of two dimensions existing in a space of four dimensions and their intersection is a point merely.

To say that this field consists of the totality of point pairs of two Neumann spheres though a useful device for the presentation of certain general arguments is not of course a representation which enables special facts like the one mentioned to be concisely recorded.


Infinite points of multiplicity greater than two occur at:
The point $E ; a: b: c:: \mathrm{I}: \frac{\mathrm{I}}{1 / 3}:-\frac{1}{1 / 3}, y=z=-\frac{1}{3}, 3 \phi-4^{\tau}=0 \phi=\infty, a=4 \beta=\infty$. Here the sheets (2)(3) (8) (9) have a common root.

The point $H ; a: b: c:: \mathrm{I}: \mathrm{I}:-\mathrm{I}, y=z=-\mathrm{I}, \phi=4, \tau=\infty, a=\infty, \beta=\infty$ where the sheets (3), (4), ( $7^{\prime}$ ), (8) unite. The line $a=0$ has six infinite roots. For the approach $a>0, \beta>0$ all six are complex, and so for $a<0, \beta<0$. If $a$ and $\beta$ have opposite signs two roots have a real approach. The triangles are complex in all cases.

The origin is an indeterminate in ( $\alpha, \beta$ ).
For $\alpha=0, \beta=0, \frac{\beta}{\alpha}=0$ there are ten zero roots. The triangles are indeterminate as $\tau$ can be assigned at will.

For $\tau=0 y=\frac{1}{8}, z=\infty$ and the triangle can be approached by way of

$$
a=\omega b+\frac{\omega+\mathrm{I}}{2 \omega+\mathrm{I}} \quad c=-b(\omega+\mathrm{I})+\frac{\omega}{2 \omega+\mathrm{I}}
$$

where $\omega^{3}=1$ and $b$ increases without limit.
For $\tau=-\mathrm{I}, y=\frac{1}{4}, z=0$. This is the point $F$ which in $(\phi, \tau)$ is represented as the line $\phi-\tau=0$.
No other triangle in the infinite set is real except $F\left(a=b=\frac{1}{2} c=0\right)$. The sheets involved are $(3),(4),(5),(7),(8),(9),(10)$, and (9) is only reached with $\tau=\infty$.

The other approaches to the origin give in some cases finite values for $\sigma$ but all the triangles are complex, as none of the sheets involved fall inside $D_{\mathrm{x}}$ in the $y, z$ plane.
$\beta=0 a \neq 0$ has ten zero roots independent of $a$, but these can only be approached in the $k, l, m$ plane with complex values and give of course complex triangles.

## I2. FINITE MULTIPLE POINTS

To determine all the finite multiple points (other than double points) a start is made from the intersections of $D_{2}$ and $T$.

The corresponding values of $\phi$ are given by
$2 \phi^{2}-\phi+\mathrm{I}=0$ for the triple points, and
$54 \phi^{3}-57 \phi^{2}+24 \phi-4=0$ for the four-points.

The approximate values for the real four-point are:

$$
\begin{array}{ll}
\phi=.4144425 \cdots & \tau=.529566 \ldots \\
a=-.99678 . & \beta=5.354^{2} .
\end{array}
$$

To find the other multiple points we write $F(\sigma, a, \beta)=0$ in the form

$$
\begin{equation*}
a=\frac{\theta_{1}}{\psi_{1}} \text { and } \frac{\partial F}{\partial \sigma}=0 \text { as } a=\frac{\theta_{2}}{\psi_{2}} \text { and } \frac{\partial^{2} F}{\partial \sigma^{2}}=0 \text { as } a=\frac{\theta_{3}}{\psi_{3}} \tag{60}
\end{equation*}
$$

Eliminating $a$ in turn between each pair of equations we obtain expressions (or), (o2), ( I 2 ) which must vanish for points of multiplicity three or higher.

If we then write $\beta=\frac{\sigma^{3}}{\tau}$ and revert to the ( $\phi, \tau$ ) plane by writing $\sigma=\phi+\mathrm{I}$, (o1), (02), (12). are of order 5 in $\phi$ and 4 in $\tau$.
(oI) represents the discriminant and so $T$ and $D_{2}$ in their $(\phi, \tau)$ form must be factors. These forms are given in equations (53) and (54).

As a fact (oI) has no other factors.

The form (I2) is:

|  | $\phi^{5}$ | $\phi^{4}$ | $\phi^{3}$ | $\phi^{2}$ | $\phi^{\mathbf{2}}$ | $\phi^{0}$ |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: |
| $\boldsymbol{\tau}^{4}$ |  |  |  |  |  | -1920 |
| $\boldsymbol{\tau}^{3}$ |  |  |  | -2712 | -280 | 1488 |
| $\boldsymbol{\tau}^{2}$ |  |  | -240 | 164 | -77 |  |
| $\boldsymbol{\tau}^{\mathrm{I}}$ |  | 792 | 948 | 60 | -18 | -78 |
| $\boldsymbol{\tau}^{0}$ | $-\boldsymbol{7}^{2}$ | -135 | -54 | 8 | -2 | -1 |

and the form (02) is:

|  | $\phi^{5}$ | $\phi 4$ | $\phi^{3}$ | $\phi^{2}$ | $\phi^{1}$ | $\phi^{0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 74 |  |  |  |  | 624 | -48 |
| $\tau^{3}$ |  |  |  | - 1584 | 112 | -32 |
| $\tau^{2}$ |  |  | 1377 | - 1 | - 3 | 16 |
| $\tau^{1}$ |  | $-462$ | - 102 | 36 | - 18 |  |
| $\tau^{0}$ | 45 | 39 | - 6 |  | I |  |

$D_{2}$ can be written

$$
16 \tau^{3}=32 \tau^{2} \phi-19 \tau \phi^{2}+3 \phi^{3}+\phi^{2}
$$

Reducing (12) and (02) to quadratics in $\tau$ by means of $D_{2}$ we have from (I2) the form I:

|  | $\phi^{5}$ | $\phi^{4}$ | $\phi^{3}$ | $\phi^{2}$ | $\phi^{\mathbf{I}}$ | $\phi^{0}$ |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: |
| $\tau^{2}$ |  | -432 | 196 | 164 | -77 |  |
| $\tau^{\mathrm{I}}$ |  | 432 | $-4^{\mathrm{I}} 4$ | 45 | -18 | 18 |
| $\tau^{0}$ | -72 | 54 | 9 | 8 | - | 2 |

and from (02) the form II:

|  | $\phi^{5}$ | $\phi 4$ | $\phi^{3}$ | $\phi^{2}$ | $\phi^{1}$ | $\phi^{0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\tau^{2}$ |  |  | $-36$ | 88 | $-67$ | 16 |
| $\tau^{\text {I }}$ |  | 54 | $-91$ | 71 | - 18 |  |
| $\tau^{0}$ | -18 | 21 | - II | - I | I |  |

Eliminating $\tau$ we have a polynomial in $\phi$ of order 16 .
Of this the factors

$$
\left(54 \phi^{3}-57 \phi^{2}+24 \phi-4\right),\left(2 \phi^{2}-\phi+1\right),\left(2 \phi^{2}+13 \phi+2\right),(\phi-2)
$$

are known, the second quadratic and the last factor entering as triple-point factors in the "equilateral" case. $2 \phi-1$ appears as a thrice repeated factor and the cubic factor is repeated. The residue is $9 \phi^{2}+\phi-1$.

These must include all triple, quadruple, etc., points on $D_{2}$.
In the elimination between (I2) and (O2) which have the form

$$
a=\frac{\theta_{1}}{\psi_{1}}=\frac{\theta_{2}}{\psi_{2}} \text { and } a=\frac{\theta_{0}}{\psi_{0}}=\frac{\theta_{2}}{\psi_{2}}
$$

common solutions of $\theta_{2}=0$ and $\psi_{2}=0$ enter which need not satisfy ( OI ):

$$
a=\frac{\theta_{0}}{\psi_{0}}=\frac{\theta_{\mathrm{I}}}{\psi_{\mathrm{I}}} .
$$

There are five such common solutions finite both ways and these give rise to the factors $(2 \phi-1)^{3}\left(9 \phi^{2}+\phi-1\right)$ after the reduction by $D_{2}$. The points they denote on $D_{2}$ are not in fact more than ordinary points on $D_{2}$, that is threefold double points.

For multiple points on $T=0$ we write $\tau=\frac{\left(6 \phi^{2}-3 \phi+1\right)}{(6 \phi-1)}$ in (12) and obtain a polynomial of order 9 in $\phi$, which has the factors

$$
\left(54 \phi^{3}-57 \phi^{2}+24 \phi-4\right)\left(2 \phi^{2}-\phi+1\right)\left(12 \phi^{2}-16 \phi+7\right)(\phi-1)(6 \phi+1) .
$$

Similar operations on (02) give as factors the same cubic and quadratic and a zero and infinite factor. These and the two linear factors of the first set not being common are extraneities and the only new points are given by $12 \phi^{2}-16 \phi+7-0$ and $T=0$. These are triple points and not on $D_{2}$.

These last points are singular points on $T=0$; being complex the usual classes are without significance.

> I3. THE DISCRIMINANT

Since $T=0$ and $D=0$ are discriminantal loci $T$ and $D$ are factors of the discriminant of $F(\sigma, \alpha, \beta)=0$. The order of $T$ in $\alpha$ is 4 , in $\beta, 6$. The order of $D$ in $\alpha$ is 3 , in $\beta, 2$. This is seen directly in the case of $D$ (15) and from the parametric form and the elimination rule for $T$ (57). To obtain a closer view the equation may be transformed by writing,

$$
\sigma=3 s, \beta=4 b, \frac{\beta(\alpha-4)}{a}=d .
$$

The result is

$$
\begin{align*}
& s^{10}-10 b s^{8}+14 b s^{7}+33 b^{2} s^{6}-94 b^{2} s^{5}+\left(61 b^{2}-36 b^{3}-b^{2} d\right) s^{4}+  \tag{65}\\
& \quad\left(156 b^{3}+b^{2} d\right) s^{3}+\left(-200 b^{3}+4 b^{3} d\right) s^{2}+\left(80 b^{3}-8 b^{3} d\right) s+4 b^{3} d=0
\end{align*}
$$

The order of the coefficients in the first derivative with respect to $s$ and a homogeneity factor, is in respect of $d$ given by the rows:

| $(0)$ | $(1)$ | $(2)$ | $(3)$ | $(4)$ | $(5)$ | $(6)$ | $(7)$ | $(8)$ | $(9)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | $\circ$ | 0 | 0 | I | I | I | I |
| 0 | 0 | 0 | 0 | 0 | I | I | I | I | I |

From this the order of the elements in Bezout's form can be readily obtained and inserted in the determinant form,

| 0 | - | 0 | 0 | 1 | I | I | I | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $\bigcirc$ | 0 | I | I | 1 | I | I | I |
| 0 | 0 | I | I | I | I | I | I | I |
| 0 | I | I | I | 1 | 1 | I | I | I |
| I | I | I | I | I | I | I | I | 1 |
| 1 | I | I | I | I | 2 | 2 | 2 | 2 |
| 1 | I | I | I | I | 2 | 2 | 2 | 2 |
| 1 | I | I | I | I | 2 | 2 | 2 | 2 |
| I | I | I | I | 1 | 2 | 2 | 2 | 2 |

The maximum order is $\mathrm{I}_{3}$ occurring in the secondary diagonal of the first 5 -square and the complete last 4 -square. A short calculation shows that this order actually occurs. The factor $d$ cannot divide the result as $d=0$ is not discriminantal for $b \neq 0$.

The order in $d$ is the order of $a$ as far as it depends on the factors $T$ and $D ; a=0$ corresponding to $d=\infty$ being represented by the defect of this order from 18 the order of the general case. As $\alpha=\infty$ is not a discriminantal locus $a^{5}$ must occur as a factor of the discriminant of $F(\sigma, a, \beta)$ $=0$. The orders of $\alpha$ in $T, D$ being 4,3 we have $4 m+3 n=13$ where $m, n$ are the exponents of $T, D$ in the discriminant. The only solution is $m=1, n=3$.

As to the powers of $\beta$ a count of order in the Bezout form for $F(\sigma, \alpha, \beta)$ gives 44 as the maximum and of these 30 can be divided from the rows and columns. Since $\beta$ enters $T . D^{3}$ to the 12 th order the exponent of $\beta$ is either $30,3 \mathrm{I}$, or 32 . The fact that for $\beta=0$ no complex roots become real or vice versa bars 3 I .

To decide between 30 and 32 it is necessary to examine the monodromie cycles for $\beta=0$.
In the neighborhood of $\beta=0$ the equation to a numerical factor is properly approximated by

$$
\left[\sigma^{3}+5 \beta\right]\left[\sigma^{3}+t_{1} \beta\right]\left[\sigma^{3}+t_{2} \beta\right][\sigma+m \beta]=0
$$

where $m=\frac{20 a}{(\alpha-4)}$ and $t_{1}, t_{2}$ are the roots of $4 t^{2}-9 t+4=0$.
Of the 45 squares of differences of the roots 9 are differences of roots belonging to the same 3 -cycle. These differences each vanish with $\beta$ and to the order $\frac{1}{3}$. The product of their squares vanishes to order 6 . There are 27 differences of roots from different 3 -cycles and these together contribute $\beta^{18}$. The 9 differences of roots of a 3 -cycle and the odd root are also of order $\frac{1}{3}$ and contribute $\beta^{6}$.

The total order to which the discriminant vanishes with $\beta$ is then 30 . It is clear that this method of determining the exponent of a discriminantal factor is general, and depends for its effectiveness only on the determination of the cycles. The only converse to the theorem is that odd exponents and odd substitutions of the roots are correlated.

The complete discriminant has the form

$$
\Delta=N \cdot a^{5} \beta^{30} T \cdot D^{3} .
$$

The determination of the numerical factor $N$ has proved impracticable.
14. the monodromie group of the equation $F(\sigma, a, \beta)=0$

To simplify the numerical work write $\sigma=3 s, a=3 a, \beta=3 b$.
The equation becomes

$$
\begin{align*}
16 a s^{10} & -120 a b s^{8}+56 a b s^{7}+297 a b^{2} s^{6}-282 a b^{2} s^{5}+61 a b^{2} s^{4}-270 a b^{3} s^{4}+36 b^{3} s^{4}+36 c a b^{3} s^{3}  \tag{66}\\
& -12 b^{3} s^{3}-150 a b^{3} s^{2}+81 a b b^{4} s^{2}-108 b^{4} s^{2}+20 a b^{3} s-54 a b^{4} s+72 b^{4} s+9 a b^{4}-12 b^{4}=0
\end{align*}
$$

We begin by placing $b=\mathrm{I}$

$$
\begin{align*}
& F(s, a, \mathrm{I})=a\left(\mathrm{I} 6 s^{10}-120 s^{8}+56 s^{7}+297 s^{6}-282 s^{5}-209 s^{4}+\right.  \tag{67}\\
& \left.360 s^{3}-69 s^{2}-34 s+9\right)+12\left(3 s^{4}-s^{3}-9 s^{2}+6 s-1\right)=0
\end{align*}
$$

For $a=1$, i.e., the cusp on $D(\alpha, \beta)=0$ the values of $s$ are given by

$$
2 s-3=0,(s-1)^{3}=0,\left(2 s^{3}+3 s-1\right)^{3}=0(\S 5)
$$

For $a=-\frac{4}{3}$ also on $D$ they are given by

$$
\left(4 s^{3}-4 s^{2}-5 s+2\right)^{2}=0, s^{3}-2 s+3=0, s+2=0 .
$$

Reference should be made to the graph, Fig. 12.
To discuss the connections of the roots at $s=\mathrm{I}, a=\mathrm{I}, b=\mathrm{I}$ we write $s=\mathrm{I}+s^{\prime}, a=\mathrm{I}+a^{\prime}$, $b=1+b^{\prime}$ and retain terms of the first three orders as this is a triple point.

We have as a proper approximation to the curve, dropping the accents,

$$
6 a-2 b+28 a b+14 a s+b s-11 b^{2}+64 a b s+45 b s^{2}-20 b^{2} s-16 s^{3}+43 a b^{2}-16 b^{3}=0 .
$$

If the origin is approached along the plane

$$
\begin{gathered}
6 a-2 b=0 \text { the }(a, s) \text { projection is } \\
17 a s-99 a^{2}+12 a^{2} s+117 a s^{2}-16 s^{3}=0 .
\end{gathered}
$$

The approximations at the origin are given by Newton's parallelogram as

$$
17 a s-99 a^{2}=0 \text { and }-16 s^{3}+17 a s=0
$$

The first branch gives a stationary root while the sccond has a parabolic factor corresponding to a possible interchange of two roots.

In fact $16 s^{2}=17 a$ and if $a$ the parameter moves round the origin in its own complex plane the complex values of $s$ each move round a half-circuit in the $s$ complex plane and are so interchanged.

We have then for this point and this approach an interchange of a pair of roots entering as part of (a cycle of) a substitution of the roots which is an element of the monodromie group.


At the same point but by an approach on $a=k b, k \neq \frac{1}{3}$ gives $8 s^{3}=(3 k-1) b$ and a circuit in the $b$ complex plane interchanges three roots cyclically and these three roots must include the former two, by the principle of the continuity of the roots, and the fact that the singular points are necessarily distinct.

The pair of roots is real for real $a$ 's and of the set of three one is real.
The same values of the parameters $a$ and $b$ give two other triple roots, one set being at $s$ $\frac{(-3+1 \sqrt{17})}{4}$. A similar treatment gives in this case also a cycle of thrce for the general approach and a transposition for approach in the tangent plane. So also for the conjugate set.

Since the tangent plancs are distinct in the three cases we have as elements of the group for a general approach a substitution of the form (234) (567) (89, 10), while for the tangent plane approaches we have the elements (23) (567) (89, 10); (234) (56) (89, 10); (234) (567) (89) respectively. The roots entcring the transpositions may be any pair of the corresponding set of three.

After the cycles at $a=1, b=1$ the monodromie path is taken along $b=1$. To determine whether any multiple points are encountered we eliminate $a$ between $F(s, a, 1)$ and its derivative
with respect to $s$. The resultant which is of order $\mathrm{I}_{3}$ in $s$ must include all finite multiple points on $b=\mathrm{I}$. Of these $s=\mathrm{I}$ and the roots of $2 s^{2}+3 s-\mathrm{I}=0$ are known, and in fact they are repeated. The remaining septimic has as a factor $4 s^{3}-4 s^{2}-5 s+2$, which gives the values at $a=-\frac{4}{5}$ on $D$. The residue is

$$
18 s^{4}-7 s^{3}-54 s^{2}+45 s-12=0 .
$$

This quartic has two complex roots and two real whose approximate values are $s=1.4598$ and $s=-1.924$. . which correspond to $a=1.47$. and $a=-1.27$. respectively. (These points are on the upper and lower branches of $T=0$.)

There being no multiple points between $a=1$ and $a=0$ we note that at $a=0$ the four real roots are
(1) $s=\mathrm{I} .57$. . which at $a=\mathrm{I}$ had the value I. 5
(2) $s=.347$. which at $a=\mathrm{I}$ had the value I
(5) $s=.333 . \quad$ which at $a=\mathrm{I}$ had the value .28
(8) $s=-\mathrm{I} .9$. which at $a=\mathrm{I}$ had the value -I .78

Since the equation is of the first order in $a$ roots can only cross at double points. This applies to complex roots as well as real for no different $a$ 's can have the same value of $s$.

At $a=0$ six roots which were complex at $a=\mathrm{I}$ become infinite.
The finite roots for $a=0$ being given by

$$
(3 s-1)\left(s^{3}-[3 s-1] b\right)=0
$$

the root marked (5) has a fixed value independent of $b$.
The cubic factor has equal roots at $b=\frac{1}{4}$ when (1) and (2) become equal and as the origin is approached with $a$ constantly zero the three roots (r) (2) (8) can be made to take part in a cycle by means of a circuit of $b$ round its own origin in the $b$ complex plane.

To determine the configuration of the roots at $a=0, s=\infty$ we notice that at $a=\mathrm{I}+a^{\prime}, a^{\prime}$ small the roots (3) (4) are conjugate complex roots of $8 s^{3}-3 a^{\prime}=0$ and as in the monodromie path from $a=\mathrm{I}$ to $a=0$ we leave the point I by decreasing $a, a^{\prime}$ is negative. The real root is then less than $I$, and the complex roots have a real part greater than 1 . We may choose (3) as the root with positive imaginary part.

In the same way it may be shown that the roots (5) (6) (7) have real parts less than I and (6) may be taken as the complex root with positive imaginary part and (5) as the real root.

For (8) (9) (го) the real part is negative and we take (8) as the real root (9) as the complex root with positive imaginary part.

As $a$ leaves the point I and decreases the paths of the six complex roots in the $s$ complex plane start in the order indicated in the diagram (Fig. I3).


Fig. 13
Now $F(s, a$, I) contains $a$ in the first order only, so no different $a$ 's can have the same value of $s$, and if the paths cross it must be at a multiple point. There are however no multiple points
for $b=1$ and $a$ between $I$ and $o$. The six complex roots then reach the infinite point of the $s$ plane in the reversed cyclic order and since by writing $s=\frac{I}{t}$ the proper approximation is

$$
4 a+9 t^{6}=0
$$

they there enter into a cycle ( $3,4,7,10,9,6$ ).
If we pass along $b=\mathrm{r}$ for $a>\mathrm{I}$ we reach a double point at $a=\mathrm{I} .47$. where $s=\mathrm{I} .4508$ is the double value.

Since at $a=1$ the root ( I ) has the value I .5 and the root (2) the value I and no double points occur in the interval we add the transposition (12) to the elements of the group.

At $a=-1.27$. . is another double point. To identify the roots here we suppose that the six-cycle at $a=0$ is so passed as to leave (3) and (10), which are opposite in the cycle, in the real positions $-\infty$ and $+\infty$ respectively, when the added element will be ( 3,8 ).

At $a=-\frac{t}{3}$ there are three pairs of equal roots of which (25) and (1, 10) are two. The third pair must be opposite in the six-cycle at $a=0$ and is then either (94) or (67). It is in fact immaterial which is taken.

As elements of the monodromie group we have:

| $A$ | $(12)$ | at $a=1.47 \ldots$ |
| :--- | :--- | :--- |
| $B_{1}$ | $(23)(567)(89,10)$ | at $a=1$ |
| $B_{2}$ | $(234)(56)(89,10)$ |  |
| $B_{3}$ | $(234)(567)(89)$ |  |
| $C$ | $(234)(567)(89,10)$ | at $a=1$ |
| $D$ | $(347,10,96)$ | at $a=0$ |
| $E$ | $(1,10)(25)(49)$ | at $a=-\frac{4}{5}$ |
| $F$ | $(38)$ | at $a=-1.27 \ldots$ |

The transposition (12) exists. By transforming it by $B_{\mathrm{x}}$ we add (13).
Transforming this by $C$ (14) is reached.
Transforming (13) by $F$ we reach (18).
Transforming (18) by $B_{\mathrm{x}}$ we reach (19) and ( $\mathrm{I}, \mathrm{ro}$ ).
Multiply $E$ by ( 1,10 ) and obtain (25) (49).
Transform (12) by this and obtain (15).
Transform (15) by $B_{1}$ and obtain (16) and (17).
We have now every ( $\mathrm{I}, n$ ) from which every single transposition and the symmetric group can be obtained.

By Jordan's theorem the group of the equation is the symmetric group.

## 15. the equation for the sides

To obtain an equation for the value of the side of a triangle with given internal anglebisectors we write $a+b+c=1$ and use ratios. We have

$$
\begin{align*}
& R=\frac{l}{k}=\frac{\mathrm{I}-2 a}{a(a-1)^{2}} \cdot \frac{b(b-1)^{2}}{\mathrm{I}-2 b} \\
& S=\frac{m}{k}=\frac{\mathrm{I}-2 a}{a(a-1)^{2}} \cdot \frac{c(c-1)^{2}}{\mathrm{I}-2 c} \tag{68}
\end{align*}
$$

Using $N=\frac{\mathrm{I}-2 a}{a(a-1)^{2}}$ as an abbreviation we obtain

$$
R+S=N \cdot \Sigma \frac{b(b-1)^{2}}{(1-2 b)}
$$

Write now $b c=\rho$ and note that $b+c=\mathrm{I}-a$ so that $b, c$ are the roots of

$$
\begin{equation*}
u^{2}-(\mathrm{x}-a) u+\rho=0 \tag{69}
\end{equation*}
$$

Expressing the symmetric function of $b, c$ in terms of the coefficients of this equation

Similarly

$$
\begin{equation*}
R+S=N \cdot \frac{\left[-a^{3}+a^{2}+\rho\left(-2 a^{2}+3 a-1\right)+4 \rho^{2}\right]}{4 \rho+2 a-1} \tag{70}
\end{equation*}
$$

$$
\begin{equation*}
R S=N^{2} \cdot \frac{\left[\rho(\rho+a)^{2}\right]}{4 \rho+2 a-1} \tag{7I}
\end{equation*}
$$

We now write $R S=p, R+S=q$, and $\rho=m-a$

$$
\begin{equation*}
p=\frac{N^{2} m^{2}(m-a)}{4 m-(2 a+1)} \quad q=\frac{\left.N\left[4 m^{2}-m\left(2 a^{2}+5 a+1\right)+a+1\right)^{2} a\right]}{4 m-(2 a+1)} \tag{2}
\end{equation*}
$$

If now $\frac{p}{q}=r$ we have a cubic and quadratic for $m$.

$$
\begin{array}{r|l|l}
4 & N m^{3}+m^{2}(-a N-4 r)+m r\left(2 a^{2}+5 a+\mathrm{r}\right)-r(a+\mathrm{r})^{2} a=0 & 0  \tag{73}\\
-m & 4 N m^{2}-m\left(N\left[2 a^{2}+5 a+\mathrm{r}\right]+4 q\right)+N\left[a(a+\mathrm{I})^{2}+q(2 a+\mathrm{r})\right]=0 & \mathrm{I}
\end{array}
$$

Using the indicated end-term multipliers whose determinant is 4 no extraneities are introduced and a second quadratic results.

If this is

$$
\begin{equation*}
A^{\prime} m^{2}+B^{\prime} m+C^{\prime}=0 \tag{74}
\end{equation*}
$$

and the previous quadratic is

$$
A m^{2}+B m+C=0
$$

the second order determinants which enter the eliminant are

$$
\begin{align*}
& A C^{\prime}-A^{\prime} C=4 q^{2}(2 a+1)+16 q r(2 a+1)-N q\left(8 a^{3}+12 a^{2}+7 a+1\right) \\
& \quad-N^{2}\left(2 a^{5}+5 a^{4}+5 a^{3}+3 a^{2}+a\right)  \tag{75}\\
& A B^{\prime}-A^{\prime} B=16 q^{2}-64 q r+4 N q\left(4 a^{2}+4 a+1\right)+N^{2}\left(4 a^{4}+8 a^{3}+a^{2}+2 a+1\right)  \tag{76}\\
& B C^{\prime}-B^{\prime} C=q^{2}(2 a+1)^{2}+2 N q(2 a+1)(a+1)^{2} a+4 q r\left(-4 a^{2}-3 a-1\right)+N^{2}(a+1)^{4} a^{2} \tag{77}
\end{align*}
$$

Collecting the coefficients of the powers and products of $N, q, r$ the eliminant takes the form

$$
\begin{align*}
\circ= & -16 q^{3} r(a) \\
& -q^{3} N(2 a+1)^{2}(2 a-1) \\
& +64 q^{2} r^{2}(a) \\
& +4 q^{2} r N(2 a+1)^{2}(a-1)  \tag{78}\\
& -q^{2} N^{2}(2 a+1)\left(6 a^{3}+5 a^{2}-a-2\right) a \\
& -q N^{3}(a+1)^{2}\left(6 a^{3}+a^{2}-2 a-1\right) a^{2} \\
& +q^{2} N^{2}(2 a+1)\left(6 a^{3}+7 a^{2}+4 a-1\right)(a-1) \\
& -N^{4}(a+1)^{4}(a-1) a^{4},
\end{align*}
$$

and writing for $N$ its value in terms of $a$, multiplying throughout by $(a-1)^{7} a^{2}$ and putting $q r=p$ we have the eliminant as an equation of the tenth order in $a$, containing two parameters $p, q$ among its coefficients. Arranged in powers and products of $p, q$ it is:

$$
\begin{aligned}
F(a: p, q)= & q^{3}(2 a+1)^{2}(2 a-1)^{2}(a-1)^{5} a \\
& -16 p q^{2}(a-1)^{7} a^{3} \\
& +64 p^{2}(a-1)^{7} a^{3} \\
& -4 p q(2 a+1)^{2}(2 a-1)(a-1)^{6} a \\
& -q^{2}(2 a+1)\left(6 a^{3}+5 a^{2}-a-2\right)(2 a-1)^{2}(a-1)^{3} a \\
& +q\left(6 a^{3}+a^{2}+2 a-1\right)(2 a-1)^{2}(a+1)^{3}(a-1) a \\
& +p(2 a+1)(2 a-1)^{2}\left(6 a^{4}+a^{3}-3 a^{2}-5 a+1\right)(a-1)^{3} \\
& -(a+1)^{4}(2 a-1)^{4} a^{2} .
\end{aligned}
$$

The result may be checked by substituting $p=\mathrm{I}, q=2$, i.e., $k=l=m$ when it reduces to give

$$
a=\infty, \infty, \frac{1}{2}, \frac{1}{3}, \frac{\left(3^{ \pm} 1 \mathrm{I} 7\right)}{4}, \frac{\left(\mathrm{I} \pm 1 \mathrm{I}_{7}\right)}{8}, \frac{(\mathrm{I} \pm 1 \mathrm{I} 7)}{8}
$$

which agrees with the previous result for this case.
To obtain $b$ as a rational function of $a$ the order of elfmination must be changed. With this order $m$ and hence $b c$ is obtained as a rational function of $a$.

If we write $\frac{R}{N}=f ; \frac{S}{N}=g$, we have two cubics in $b$ :

$$
\begin{aligned}
& b^{3}-2 b^{2}+b(1+2 f)-f=0 \\
& b^{3}+b^{2}(3 a-1)+b\left(3 a^{2}-2 a+2 g\right) c+\left(a^{3}-a^{2}+2 a g-g\right)=0
\end{aligned}
$$

by carrying the elimination to the penultimate step by the end-term method we have

$$
\begin{aligned}
& b\left[-2 f^{2}+2 g^{2}(2 a-1)-4 f g(a-1)+g\left(8 a^{3}+3 a^{2}-2 a-1\right)-2 f(a+1)^{2} a+\left(a^{2}-1\right)(3 a+1) a^{2}\right] \\
& +\left[g^{2}(2 a-1)^{2}+2 f g\left(-6 a^{2}-1\right)+f^{2}+g(2 a-1)\left(2 a^{3}-2 a^{2}-3 a-1\right)+f\left(-6 a^{4}-3 a^{3}+3 a^{2}+2 a\right)\right. \\
& \left.\left(a^{2}-1\right)\left(a^{2}-2 a-1\right) a^{2}\right]=0 .
\end{aligned}
$$

In this expression

$$
f=\frac{l}{k}\left[\frac{(a-\mathrm{I})^{2} a}{(-2 a+\mathrm{I})}\right]: g=\frac{m}{k}\left[\frac{(a-\mathrm{I})^{2} a}{(-2 a+\mathrm{I})}\right]
$$

making these substitutions and dividing by $\frac{a(a-1)}{(2 a-1)^{2}}$ we have unless $a=\mathrm{I}, \frac{1}{2}$, or o

$$
\left.\begin{array}{c}
b\left[-2 l^{2}(a-\mathrm{I})^{3} a-4 \operatorname{lm}(a-\mathrm{I})^{4} a+2 m^{2}(a-\mathrm{I})^{3}(2 a-\mathrm{I}) a+2 l k(a+\mathrm{I})^{2}(a-\mathrm{I})(2 a-\mathrm{I}) a-\right. \\
\left.\quad m k(a-\mathrm{I})(2 a-\mathrm{I})\left(8 a^{3}+3 a^{2}-2 a-\mathrm{I}\right)+k^{2}(a+\mathrm{I})(3 a+\mathrm{I})(2 a-\mathrm{I})^{2} a\right] \\
+\left[l^{2}(a-\mathrm{I})^{3} a+2 l m(a-\mathrm{I})^{3}\left(-6 a^{2}-\mathrm{I}\right) a+l k(a-\mathrm{I})(2 a-\mathrm{I})\left(6 a^{3}+3 a^{2}-3 a-2\right) a+k^{2}(a+\mathrm{I})\right.  \tag{80}\\
\left.\quad\left(a^{2}-2 a-\mathrm{I}\right)(2 a-\mathrm{I})^{2} a+m^{2}(a-\mathrm{I})^{3}(2 a-\mathrm{I})^{2} a-m k(2 a-\mathrm{I})^{2}(a-\mathrm{I})\left(2 a^{3}-2 a^{2}-3^{a} a-\mathrm{I}\right)\right]=0
\end{array}\right\}
$$

In obtaining approximate solutions substitution in this expression is more laborious than the solution of a cubic. Hence on this ground the solution by the $\sigma$ chain (II) and (3) is preferable. In practice however the method of trial on the equations (22) is more expeditious than either.

Theoretically we note that the sides involve merely an irrationality of the tenth degree and the cubic which occurs in the $\sigma$ chain is not necessary but convenient. The cubic irrationality is then not accessory in the technical sense.
$F(a: p, q)$ is of course unsymmetric in $(k, l, m)$ and as $p=\frac{l m}{k^{2}}, q=\frac{(l+m)}{k}$, it can be denoted by $F(a, k: l, m)$ and is associated with two other equations by cyclic change.

That these equations which have the same form are all irreducible can be established thus:
Suppose they reduce in ( $k, l, m$ ), choose a corresponding set of factors, and from them form the cquation whose roots are $\sigma$ expressed as a symmetric function of $a, b, c$.

The coefficients of this equation will be symmetric functions of $k, l, m$ of order zero and hence rational in $\alpha, \beta$.

For

$$
\begin{equation*}
a=\frac{(k+l+m)^{2}}{k l+l m+m k} \text { and } \beta=\frac{(k+l+m)^{3}}{k l m} \tag{7}
\end{equation*}
$$

and writing $k+l+m=P, k l+l m+m k=Q, k l m=R$ every coefficient when the denominators are cleared will be the sum of multiples of such terms as $R^{\xi} Q^{\eta} P^{\zeta}$ where on account of the homogeneity $3 \xi+2 \eta+\zeta=n$.

On dividing such terms by $P^{n}$ we obtain terms of the form

$$
\frac{R^{\xi} Q^{\eta}}{P^{\eta-\zeta}}=\frac{R^{\xi} Q^{\eta}}{P^{3 \xi}+2 \eta}=\beta^{-\xi} a^{-\eta}
$$

The equation will be satisfied by the corresponding values of $\sigma$ if the equations in $a, b, c$ are satisfied and will be of an order less than ro. But $F(\sigma, \alpha, \beta)$ which will be the result of carrying out the above process on all the factors of $F(a, k: l, m)$, etc., does not reduce, which contradicts the hypothesis of these equations being reducible.

## I6. THE MONODROMIE GROUP OF THE EQUATION FOR THE SIDES

The form in which the equation emerges from the elimination (79) renders the determination of critical points with the corresponding binomial approximations and cycles of the roots easy.

We deal first with the values $p=\infty, q=\infty$. If $p=\lambda q$ and $\lambda$ is finite the common factors of the terms of highest order in $q$ are $(a-\mathrm{I})^{5} a$. For $a=\mathrm{I}=n, q=\frac{\mathrm{I}}{\kappa}$ the terms to be considered are of orders $n^{5}, \kappa n^{3}, \kappa^{2} n, \kappa^{3}$.

The first three give 2 two-cycles and the last pair a stationary root at $a=1$, there is a single root at $a=0$ and four other roots dependent on $\lambda$ and in general distinct.

This approach to the point $p=\infty, q=\infty$ gives then an element of the group of the type ( 12 ) (34).

If we write $4 p=q^{2}$ and then $q=\frac{\mathrm{I}}{\kappa}$ we get the same state of things at $a=\mathrm{I}$, but at $a=0$ there is a two-cycle, the other roots are an odd one at $a=\frac{1}{2}$ and a pair not forming a cycle at $a=-\frac{1}{2}$.

Since on the Neumann spheres for $p, q$ the points used are as close as we please we may identify the totality of roots at $a=\mathrm{I}$ in the two cases. For a real approach the two pairs of roots at $a=\mathrm{I}$ are as near as we please two pairs with equal values and opposite sign and as this character is the same for all finite values of $\lambda$ no discriminantal branch is crossed and the pairing may be identified in the two approaches. By the product of the two substitutions a single transposition is obtained.

For the approach with $\lambda=0$, or by writing $q=0$ and then $p=\frac{1}{\pi}$ we have for $a-\mathrm{r}=n$ a binomial approximation of the type $n^{7}=\pi^{2}$ giving a seven-cycle, and for $a$ near o a three-cycle.

By the continuity of the roots and the approximate equality of $p, q$ in the cases $\lambda=0$ and
$\lambda$ finite but small we may identify the two roots which enter the single transposition with two of those in the three-cycle, and conclude that the seven-cycle contains the four roots which entered the pair of two-cycles in the other approaches.

We then have as elements of the group

$$
A:(a \beta \gamma \delta \epsilon \zeta \eta)(\theta \sigma \kappa) \text { and } B:(\theta \sigma)
$$

At the point $q=\lambda p=0$ there is a four-cycle at $a=-1$, a second four-cycle at $a=\frac{1}{2}$, and a two-cycle at $a=0$. This gives the element

$$
C:(a b c d)(e f g h)(i j)
$$

From combinations of $A^{7}$ and $B$ the subgroup $G_{6}(\theta \sigma \kappa)$ is generated and one of these roots must be connected by $C$ with some root in the set of seven in $A^{3}$. By using the powers of $A^{3}$ as transformers every single transposition of the ten roots can be produced and hence the group $G_{10}$ !

The monodromie group being the symmetric group by Jordan's theorem the algebraic group is also the symmetric group.

## 17. the reduction of the equation for $\sigma$ in the case of equal bisectors

In this case two angle-bisectors are equal. If $K=L$ and $\frac{K^{2}}{M^{2}}=m$

$$
\alpha=\frac{(m+2)^{2}}{2 m+1} \quad \beta=\frac{(m+2)^{3}}{m}
$$

Taking $m$ as the single parameter and substituting for $\alpha, \beta$ and writing $\sigma=\xi(m+2)$ the equation when divided by $(m+2)^{\text {ro }}$ becomes

$$
\begin{align*}
& \left.16 m^{3} \xi^{10} \dot{-} 40 m^{2}(m+2) \xi^{8}+56 m^{2} \xi_{7}+33 m(m+2)\right)^{2} \xi^{6}-94 m(m+2) \xi^{5}-\left(10 m^{3}+52 m^{2}+39 m+7^{2}\right) \xi^{4}  \tag{8I}\\
& \quad+\left(40 m^{2}+152 m+156\right) \xi^{\xi^{3}}+\left(m^{3}-62 m-116\right) \xi^{2}+\left(-m^{2}+4 m+36\right) \xi+(m-4)=0
\end{align*}
$$

Since in general $\beta=\frac{\sigma^{3}}{\tau}, \sigma=m+2, \tau=m$ is suggested as a solution. This gives $\xi=1$ which satisfies the equation and dividing out the factor we have:

$$
\left.\begin{array}{l}
16 m^{3} \xi^{9}+16 m^{3} \xi^{8}+\left(-24 m^{3}-80 m^{2}\right) \xi^{7}+\left(-24 m^{3}-24 m^{2}\right) \xi^{6}+\left(9 m^{3}+108 m^{2}+132 m\right) \xi^{5} \\
+\left(9 m m^{3}+14 m^{2}-56 m\right) \xi^{4}+\left(-m^{3}-38 m^{2}-95 m-72\right) \xi^{3}+\left(-m^{3}+2 m^{2}+57 m+84\right) \xi^{2}  \tag{82}\\
+\left(2 m^{2}-5 m-32\right) \xi+(-m+4)=0
\end{array}\right\}
$$

This must be of the form

$$
\left(A \xi^{3}+B \xi^{2}+C \xi^{\xi}+D\right)^{2}\left(\alpha \xi^{3}+\beta \hat{\xi}^{2}+\gamma \xi+\delta\right)=0
$$

where the coefficients are polynomials in $m$. We see at once that $D=1$. Recalling the special cases

$$
\begin{aligned}
& a=3, \beta=27, m=1 \\
& a=4, \beta=54, m=4
\end{aligned}
$$

where the second factor is

$$
\begin{array}{r}
4 \xi^{3}+0 \xi^{2}-11 \xi+3 \\
16 \xi^{\xi_{3}}+0 \xi^{\xi^{2}}-20 \xi+0
\end{array}
$$

respectively, and seeing that the polynomials cannot be of higher than the first order in $m$ we try
which divides with quotient

$$
\begin{equation*}
4 m \xi^{3}+o \xi^{2}-(3 m+8) \xi-(m-4) \tag{83}
\end{equation*}
$$

$\left[2 m \xi^{3}+m \xi^{2}-(m+3) \xi+1\right]^{2}$

The solutions for $D(a, \beta)=0$ are then
I) $\sigma=2+m, \tau=m$ : which gives $y=\infty, z=\infty$, and an infinite triangle whose sides are $\left[\frac{1}{2},+\infty,-\infty+\frac{1}{2}\right]$ a solution for every $m$.

This has been discussed in the case of the equilateral triangle.
2) The factor $4 m \xi^{3}-(3 m+8) \xi-(m-4)$ gives three isosceles triangles. Its discriminant being

$$
\begin{equation*}
\frac{2^{8}}{3^{3}} m\left(27 m^{2}+9 m+32\right) \tag{85}
\end{equation*}
$$

there are three real roots for positive $m$ 's, one real root for negative $m$ 's.
The distribution of $m$ over $D(a, \beta)$ is as follows:

$$
\begin{array}{ll}
0<m<\mathrm{I} & \text { from } a=4, \beta=\infty \text { to the cusp } \\
\mathrm{I}<m & \text { from the cusp to } a==\infty, \beta=\infty \\
0>m>-\frac{1}{2} & \text { the hyperbolic branch in the second quadrant } \\
-\frac{1}{2}>m>-2 & \text { the branch in the third quadrant } \\
-2>m & \text { the parabolic branch in the fourth quadrant }
\end{array}
$$

The origin is $m=-2$, the asymptotic points $m=0, m=-\frac{1}{2}$.
For $0<m<$ I the three real roots of this factor are those called $1,5,10$.
For $m>\mathrm{I}$ they are $\mathrm{I}, 7,8$.
The root ( I ) gives an isosceles triangle with internal bisectors equal.
The root (5) gives an impossible triangle with real bisectors.
The root (10) gives a real possible isosceles triangle with equal external bisectors at the base which is the smallest side. The larger bisector is internal.

The root (7) gives an impossible triangle, while (8) has a real isosceles triangle, the equal bisectors being external, the smaller one internal.

For negative $m$ 's if $m>-\frac{1}{2}$ the point representing the real solution (which gives a complex triangle) falls in $X X X_{2}$. In the other cases the triangle is real and impossible, for $-2<m<-\frac{1}{2}$ the point falls on the boundary of $\oplus_{2}$ and $X X X_{4}$, for $m<-2$ on the boundary of $\oplus_{1}$ and $X X X_{3}$.
3) In the case of the squared cubic factor the discriminant is

$$
\begin{equation*}
\frac{m}{27}\left[9 m^{3}+38 m^{2}+9 m+216\right] \tag{86}
\end{equation*}
$$

For $m>0$ there are always three real roots. The cubic factor of the discriminant has one real root only,

$$
m_{1}=-4.987 \ldots, \alpha=-.997 \ldots, \beta=5.354 \ldots \quad(\text { See } \S \text { 12.) }
$$

For $m<m_{\mathrm{I}}$ there are three real solutions, for $0>m>m_{\mathrm{I}}$ only one. Along the latter part of $D(a, \beta)$ two pairs of complex roots become equal. The point $m_{\mathrm{r}}$ is the crossing point of $T, D_{2}$ in the $\phi, \tau$ plane and a point of tangency of $T$ and $D$ in the $a, \beta$ plane: the real fourpoint. For $0<m<1$ the solutions fall in $(2,3),(6,7),(8,9)$.

The $(2,3)$ solution is a real possible triangle with one of the equal smaller bisectors internal. The $(6,7)$ solution is impossible. The $(8,9)$ solution like the $(2,3)$ has one of the equal smaller bisectors internal but the smallest side is much smaller, the opposite angle being always $<20^{\circ}$.

For $\mathrm{I}<m<\infty$ the solutions fall in $(3,4),(5,6),(9,10)$.
The $(3,4)$ case has the less bisector internal. The $(5,6)$ triangle is impossible.
The ( 9,10 ) case has one of the larger bisectors internal.

In the case of isosceles triangles it is obviously unnecessary to solve two cubics. The cubic for the side $a$ is

$$
4(m-4) a^{3}-(9 m-16) a^{2}+2(3 m-2) a-m=0
$$

whose discriminant

$$
\begin{equation*}
\frac{m}{27}\left(27 m^{2}+9 m+32\right) \tag{87}
\end{equation*}
$$

is algebraically identical with the $\xi$ discriminant involved.
In this case $b=a, c=1-2 a$ completes the solution.

$$
\left[\text { For } K=L, a=b, m M^{2}=K^{2}\right]
$$

I8. Reduction of the equation for the side in the case of equal bisectors
If $l=m, q=\frac{2 m}{k}, p=\frac{m^{2}}{k^{2}}$.
Writing $q=2 R, p=R^{2}$ in the equation the coefficient of $R^{4}$ vanishes identically and (2a-1) is a factor of the remaining terms.

The residue is

$$
\begin{array}{r}
8 R^{3}(2 a+1)^{2}(a-1)^{5} a^{2}+R^{2}(2 a+1)(2 a-1)(a-1)^{3}\left(-18 a^{4}-19 a^{3}+a^{2}+3 a+1\right)  \tag{88}\\
+2 R(2 a-1)^{2}(a+1)_{2}(a-1)\left(6 a^{3}+a^{2}-2 a-1\right)-(a+1)^{4}(2 a-1)^{3} a^{2}=0
\end{array}
$$

The case of isosceles triangles gives

$$
8 R(a-1) a^{2}-(2 a-1)(a+1)^{2}=0 \text { as a factor. }
$$

The other factor is $R^{2}(2 a+1)^{2}(a-1)^{4}-2 R(2 a+1)(2 a-1)(a-1)^{2}(a+1) a+(a+1)^{2}$ $(2 a-1)^{2} a^{2}$. This is, as it should be, a square.

For $l=m$ we have then for $a$

1) $a=\frac{1}{2}$ giving the infinite triangle discussed in ( $\S 5$ )
2) $k(a+1)^{2}(2 a-1)-8 m(a-1) a^{2}=0$
for the three isosceles triangles, $b=c=\frac{(1-a)}{2}$.
3) Three triangles each a double solution from

$$
\begin{equation*}
k(a+1)(2 a-1) a-m(2 a+1)(a-1)^{2}=0 \tag{90}
\end{equation*}
$$

In the last case after finding $a$ we have a quadratic for $b$ and $c$.
For $b+c={ }_{\mathrm{I}}-a$ and since $l_{b}=m_{c} ; D_{2}(b, c)=0$

$$
\begin{equation*}
(b+c)^{2}-2 b c(b+c)+3 b c-2(b+c)+1=0 \tag{40}
\end{equation*}
$$

from which

$$
b c=-\frac{a^{2}}{2 a+1} \text { and with } b+c=1-a
$$

a quadratic for $b, c$.
Hence the solution of a cubic and quadratic is sufficient.
It will now be shown that no simpler solution exists for a general value of $\frac{k}{l}$.
If we discuss the corresponding factor for $F(a, k: l, m)$ for $k=l$ the method of elimination (§ Ig) obviously fails to distinguish $a$ and $b$ in any way and we get a sextic factor for the $D_{2}$ case $k=l, a \neq b$.


This factor gives all the $a$ 's and $b$ 's needed to make up the three triangles involved, the connection of the pairs being determined by

$$
D_{2}(a, b) \equiv(a+b)^{2}-2 a b(a+b)+3 a b-2(a+b)+1=0
$$

This rational relation holding for three pairs of the loots the group of the sextic reduces. In fact we have shown how by solving a cubic for $c$ to find $a$ and $b$ by solving a quadratic. If the sextic is irreducible, the group is transitive and cannot further reduce than is indicated by this solution.

The sextic, obtained by a process similar to that which afforded the corresponding cubic, is

$$
\begin{equation*}
m^{2}\left[(a-1)^{3}\left(8 a^{2}-4 a-1\right) a\right]+m l\left[(a-1)\left(-8 a^{4}+4 a^{3}-4 a^{2}+5 a-1\right) a\right]+l^{2}\left(3 a^{2}-1\right)^{2} \tag{9I}
\end{equation*}
$$

The other factor is

$$
\begin{equation*}
m(4 a-1)(a-1)^{2}-4 l(2 a-1)^{2} a \tag{92}
\end{equation*}
$$

the tenth root being infinite.
The irreducibility of the sextic is easily established.
If the roots are paired as $(1,2)(3,4)(5,6)$ we may write the rational relation $D_{2}(\mathrm{I}, 2)=0$ shortly as (I2) $=0$.
(The function (12) $+(34)+(56)=0$ is in $G_{48}$ and distinct from its conjugates.)
The group is as in general in the case where a general cubic with a parameter and a quadratic give rise to a sextic on elimination $G_{48}$ generated by the substitutions (12) : (135)(246): (13)(24).

As the Galois resolvent may be taken the equation of degree 48 rational in $m$ which has for roots

$$
a_{5}-b_{\mathrm{t}}+\omega\left(a_{2}-b_{2}\right)+\omega^{2}\left(a_{3}-b_{3}\right) \text { and its conjugates. }
$$

Each root may be rationally expressed in terms of any one of these.
It is interesting to note what happens to the rational expression of $b$ in terms of $a$ in general valid for the tenth-degree equation for $a$.

In this case $k=l$ if we solve $F(c, m: k, l)=0$ for $c$, we have a cubic for $c$ if $a \neq b$.
The rational expression (80) for $a$ becomes

$$
\begin{equation*}
a=\frac{\left[4 c k-m(c-1)\left(c^{2}-2 c-1\right)\right]}{m(c-1)(3 c+1)} \times \frac{(2 c \pm 1) k+m(c+1) c^{2}}{(2 c+1) k+m(c+1) c^{2}} \times \frac{(2 c-1)^{2}}{(2 c-1)^{2}} \tag{93}
\end{equation*}
$$

for $c=\frac{1}{2}, a$ is indeterminate, the limiting value leading to the infinite triangle previously discussed (§5).

For the isosceles case

$$
\frac{k}{m}=-\frac{(c-1)(c+1)^{2}}{8}
$$

whence $a=b=\frac{(\mathrm{I}-c)}{2}$.
For the case $a \neq b$ however

$$
\frac{k}{m}=-\frac{(c+1) c^{2}}{(2 c+1)}
$$

and the expression for $a$ becomes indeterminate. The limiting value gives

$$
a=-\frac{\left(2 c^{3}-c^{2}+1\right)}{(2 c+1)(c-1)},
$$









$$
\operatorname{SURFACE} \mathrm{F}(\sigma, a, \beta)=0
$$

whence

$$
b=\frac{2 c^{2}}{(2 c+1)(c-1)} \text { and } a b=-\frac{2 c^{2}\left(2 c^{3}-c^{2}+1\right)}{(2 c+1)^{2}(c-1)^{2}} .
$$

This value for $a b$ is however inconsistent with $D_{2}(a, b)=0$ which gives

$$
a b=-\frac{c^{2}}{(2 c+1)}
$$

'Hence the rational expression fails as was to be foreseen from the group theory.

$$
\text { 19. THE SURFACE } F(\sigma, \alpha, \beta)=0
$$

Since $a$ is single valued we take the $\alpha$ axis vertical. $\quad \beta$ and $\sigma$ to the right and up respectively in the plane of the diagram. (Fig. 14.)

This shows the ridge lines $T=0, D_{2}=0$ and the lines $D_{\mathrm{r}}=0$ and $\phi-\tau-\mathrm{I}=0$ marked $\phi$ where the discriminantal cylinder has an ordinary intersection with the surface.

The projection of the asymptotic cylinder $\alpha=\infty$ on the $\beta, \sigma$ plane is marked. The crosssections give a descriptive idea of the surface. (Fig. I5.)

Drawing to scale is unfortunately impossible as the small loop of the asymptotic cylinder, only extends to $\beta=.016$. . and the real region commences at $\beta=27$ where the ridge lines have a triple tangent.

## I. THE EXTERNAL PROBLEM

The formulas for the external bisectors being

$$
\begin{aligned}
K^{2} & =\frac{(a-b+c)(a+b-c) b c}{(b-c)^{2}} \\
L^{2} & =\frac{(-a+b+c)(a+b-c) c a}{(c-a)} \\
M^{2} & =\frac{(-a+b+c)(a-b+c) a b}{(a-b)^{2}}
\end{aligned}
$$

as in the internal case we use ratios and write

$$
\begin{gathered}
K^{2}: L^{2}: M^{2}:: \frac{1}{k}: \frac{1}{l}: \frac{1}{m} \\
a+b+c=1 \\
a=\frac{(k+l+m)^{2}}{k l+l m+m k} \quad \beta=\frac{(k+l+m)^{3}}{k l m} .
\end{gathered}
$$

Expressing $a, \beta$ in terms of $x, y, z$ elementary symmetric functions of the sides and writing $x=1$ we have

$$
\begin{align*}
& \alpha=\frac{\left(4 y^{2}-y-3 z\right)^{2}}{4 y^{4}-y^{3}-6 y^{2} z+9 z^{2}-3 y z+z} \\
& \beta=\frac{-\left(4 y^{2}-y-3 z\right)^{3}}{z(4 y-8 z-1)\left(-4 y^{3}+y^{2}+18 y z-2 z^{2}-4 z\right)} \tag{2}
\end{align*}
$$

The cubic expression in the denominator of $\beta$ is

$$
P^{2} \equiv[(a-b)(b-c)(c-a)]^{2}
$$

the discriminant of the cubic whose roots are the sides.
We notice also that

$$
\begin{equation*}
a-4=\frac{p^{2}}{4 y^{4}-y^{3}-6 y^{2} z+9 z^{2}-3 y z+z} \tag{3}
\end{equation*}
$$

Hence all isosceles triangles have $\alpha=4, \beta=\infty$.
Points on $\alpha=4$ for which $\beta \neq \infty$ are reached only in the $y z$ plane along $4 y^{2}-3 z=0$ as limits for $y=\infty$, which gives infinite sides with complex approach.

As in the case of the internal problem it is convenient to eliminate in two ways.

Writing

## 2. THE FIRST ELIMINATION

$$
\begin{align*}
& 3 y=\frac{1}{(\rho-\sigma+1)} \\
& 3 z=\frac{\left(4 y^{2}-y\right) \sigma}{\rho} \tag{4}
\end{align*} 60
$$

we obtain

$$
\begin{aligned}
\frac{4-\alpha}{a} & =\frac{3(4 \rho \sigma+3 \sigma+\rho+1)}{3^{\sigma-3 \rho+1}} \\
\frac{\beta(4-a)}{81 a} & =\frac{\rho(\sigma-\rho)(\rho-\sigma+1)^{2}}{\sigma\left(-9 \rho^{2}+9 \rho \sigma-9 \rho+8 \sigma\right)\left(3^{\sigma}-3 \rho+1\right)}
\end{aligned}
$$

From the first of these

$$
3 \rho=\frac{(3 \sigma+1)(1-a)}{(1+a \sigma)},
$$

and by substitution in the second

$$
\begin{array}{r}
\beta(a-4) \sigma(\sigma+1)^{2}(1+a \sigma)\left[-a^{2}(\sigma+1)^{2}+a\left(9 \sigma^{2}+10 \sigma+5\right)-4\right]  \tag{6}\\
\cdot=(\alpha-1)\left[\alpha\left(3 \sigma^{2}+3 \sigma+1\right)-1\right]\left[\alpha\left(3 \sigma^{2}+1\right)-4\right)^{2}
\end{array}
$$

This equation of the first order in $\beta$ with no common factor of the coefficients of $\beta^{\mathrm{x}}$ and $\beta^{\circ}$ is necessarily irreducible in $R(\alpha, \beta)$.

Since all subsequent operations must depend either for their necessity or their effectiveness on the nature of the group of the equation it is proper to determine this in advance if possible.

## 3. THE GROUP OF THE EQUATION

For $a=0$ the equation (6) reduces to $\beta \sigma(\sigma+1)^{2}=1$ which has a binomial approximation $s^{3}=\beta$ for $\sigma=\frac{\mathrm{I}}{\mathrm{s}}$ at $s=0, \beta=0$, and hence a cyclic substitution of order three among the roots. At $\beta=-\frac{27}{4}, \sigma=-\frac{1}{3}$ a double root occurs giving a two-cycle and establishing the symmetric group on these three roots as a subgroup of the monodromie group.

For $\beta=0$ the equation reduces having a pair of squared factors. Further equalities occur at $a=0, \mathbf{I}, 3,4$. These facts may be represented in a graph where the upper curve is double (Fig. 16).

For $\alpha \neq 0,1,3,4$ a proper approximation is of the form $\kappa \beta=\left(\sigma-\sigma_{I}\right)^{2}$ where $\sigma_{I}$ is one of the doubled roots. The two values of $\sigma_{1}$ lead to a substitution of the form (12) (45). At $\alpha=1$, $\sigma=-\mathrm{I}$ and at $\alpha=3 \sigma=-\frac{1}{3}$, the crossing points of the curves, no three-cycles occur, but at $\alpha=4$ the approximate forms are $3\left(12 s^{2}+a\right)^{2}=-4 \beta a s^{2}$ for $\sigma=0+s$ and $\alpha=4+a$, and also $48 s^{2}+a=0$ at $\sigma=-\frac{1}{2}+s$. These give a substitution of the form (14) (25) (36). The two substitutions may be denoted by $U$ and $V$ respectively.

For $a=3, \beta=27$, the equal bisector point, the equation reduces to a perfect sixth power. The approximation is

$$
2916 s^{6}-10 a \text { where } \sigma=-\frac{1}{3}+s, a=3+a, \beta=27 .
$$

At this point we have a six-cycle.
Without further specifying the identity of the roots we may now prove that the group is the symmetric group on six letters.

Assuming that the six-cycle is ( 123456 ) the three-cycle falls in this either with an adjacent pair or in alternate positions. That is either 1,2 , occur in $(a ; b, c)$ or $(a, b, c)$ is (135).

In the first case transforming (12) by the six-cycle we get (23), (34), (45), (56), (61) and
compounding with (I2) successively (I3), (14), (I5) and having now all single transpositions, all the substitutions of the symmetric group follow.

In the second case we have

$$
S=(123456): T=(135) \text { and also (13) },(35),(15)
$$

By using $S$ as a transformer we add $(246):(24):(46):(62)$ and the roots fall so far into two disconnected sets. The cube of $S$ is (14) (25) (36) and $V$ is of the same form. If the two are not identical $V$ either keeps the same division or connects the odd and even. In the latter case the transforms of the transpositions already at hand give all transpositions, in the former cases $U$. can be used as a transformer on $V$ and the same result is reached, namely the symmetric group. As in the previous problem the algebraic group is also the symmetric group.


The equation in $\sigma$, although convenient for determining the irreducibility and the group, is under disadvantages in other respects. The expressions for $y, z$ are rather complicated and fail to give determinate values in rather a large number of special cases. These occur for the following values:
$a=0 \beta \neq 0$ when three roots are infinite but $\rho$ has the factor $\mathrm{I}+a \sigma$ in its denominator and is indeterminate.
$a=\mathrm{I} \beta \neq 0$ when $\sigma=-\mathrm{I}$ is a fourfold root and $\rho$ is indeterminate.
$a=3 \quad$ when $\sigma=-\frac{1}{3}$ is a threefold root, $\rho$ indeterminate.
It is not necessary to conclude that $a(a-1)(a-3)$ is a factor of the discriminant, for the definite values of $\sigma$ which arise point out a discriminantal point rather than a discriminantal locus.

To avoid these things which cannot all be successfully dealt with by limits a second method of elimination is convenient.

## Writing

4. THE SECOND ELIMINATION

$$
\begin{equation*}
\frac{\beta(a-4)}{a}=B: \frac{(1-\alpha)}{a}=A \tag{7}
\end{equation*}
$$

we have

$$
\begin{gather*}
\frac{y-4 y^{2}+3 z}{z(4 y-8 z-1)}=B  \tag{8}\\
\frac{(3 y-1)\left[z(6 y-1)-y^{2}(4 y-1)\right]}{\left(3 z-4 y^{2}+y\right)^{2}}=A
\end{gather*}
$$

These are quadratics in $z$. The elimination is simplified by writing $z=\left(4 y^{2}-y\right) t$ and eliminating $t$, after division by $\left(4 y^{2}-y\right)$. The values $y=0, y=\frac{1}{4}$ do not in general give solutions, with $z=0$.

Arranged in powers and products of $A, B$ the result is

$$
\begin{align*}
F(y, A, B) \equiv & A^{2} B^{2}[4 y-1]^{3}[8 y-3]^{2} y \\
& +A B^{2}[4 y-1]^{2}[3 y-1]\left[128 y^{2}-96 y+17\right] y  \tag{9}\\
& -9 A B[4 y-1]^{3}[3 y-1] \\
& +8 B^{2}[4 y-1]^{2}[3 y-1]^{2}[2 y-1] y \\
& -27 A[3 y-1]^{2} \\
& -8 B \quad[3 y-1]^{3}[6 y-1]=0
\end{align*}
$$

For $a=3, \beta=27: A=-\frac{2}{3}, B=-9$, this reduces to

$$
36(2 y-1)^{6}=0
$$

giving a unique complex triangle with equal external bisectors.
Here $y=\frac{1}{2}, z=1 \frac{1}{2}$, and

$$
a: b: c=2+{ }^{3} \sqrt{2}-3^{3} V^{4}: 2+\omega^{3} v^{2}-\omega^{2} v^{-} / \overline{4}: 2+\omega^{2} \mathbf{v}^{-} \overline{2}-\omega^{3} \mathbf{v}^{4}
$$

where $\omega=\frac{(-\mathrm{I}+\sqrt{-3})}{2}$.
That this equation is irreducible and has the general group follows from the fact that $y$ is a rational function of $\sigma, u, \beta$, while $A, B$ are rational functions of $\alpha, \beta$.

Expressed as a rational function of $(y, A, B), z$ is given by

$$
\begin{equation*}
z=\frac{\left(4 y^{2}-y\right)[8 A B(4 y-1) y+8 B(3 y-1) y+9 A]}{3 A B\left[64 y^{2}-28 y+3\right]+8 B\left[18 y^{2}-9 y+1\right]+27 A} \tag{I0}
\end{equation*}
$$

This is the form obtained at the penultimate step of the elimination and has the disadvantage of being indeterminate for $27 A=B, y=\frac{1}{4}$ in which case the form

$$
\begin{equation*}
8 z=4 y-1+\frac{3 A(4 y-1)(8 y-3)+8(3 y-1)^{2}}{8 A B\left(4 y^{2}-y\right)+8 B\left(3 y^{2}-y\right)+9 A} \tag{II}
\end{equation*}
$$

or $z=-\frac{1}{72 A}$ for the special values, may be used. This is identical in virtue of $F(y, A, B)=0$. The rational integral expression for $z$ in $y, A, B$ appears to be too complicated for use.

## 5. MULTIPLE ROOTS

The method is that of the internal case (§ IO). If $a+b+c=1$ and $z=a b c$, we have,

$$
\begin{equation*}
k z=\frac{a\left(a^{3}-2 a^{2}+a-4 z\right)}{2 a^{2}-a+4 z} \tag{12}
\end{equation*}
$$

$z$ taking the place of $t$ as a parameter in each equation of the set of three but not being a proportionality factor as in the internal case, the algebra becomes a little more complicated.

For variations near a solution we have

$$
\begin{align*}
& d a\left[-4 a^{3}+6 a^{2}-2 a(\mathrm{I}-2 k z)+z(4-k)\right] \\
+ & d k\left[2 a^{2}-a+4 z\right] z  \tag{I3}\\
+ & d z\left[2 a^{2} k+a(4-k)+8 k z\right]=0
\end{align*}
$$

At ordinary points there are three equations of the form

$$
\begin{align*}
d a & =A d z+B d k \\
d b & =C d z+D d l  \tag{14}\\
d c & =E d z+F d m
\end{align*}
$$

As in the internal case we conclude that

$$
A+C+E \equiv T
$$

is a factor of the discriminant and also that $k=l, D_{2}=0$ gives rise to double roots. Whether or not $D_{2}$ recurs as in case of the internal problem is more troublesome to determine on account of the complexity of the expressions, and the question is postponed.

As before we investigate these discriminantal factors

$$
\begin{equation*}
T \equiv \Sigma \frac{a\left[2 a^{5}-5 a^{4}+(8 z+4) a^{3}-(16 z+1) a^{2}+8 a z-16 z^{2}\right]}{-4 a^{5}+7 a^{4}+16 a^{3} z-4 a^{3}+(16 z+1) a^{2}-8 a z+16 z^{2}} \tag{15}
\end{equation*}
$$

Expressing the summand as far as possible in terms of $y, z$ by means of

$$
T \equiv \Sigma \frac{a^{3}-a^{2}+a y-z=0}{\left[\left(2 y^{2}-8 y z-z\right) a^{2}+\left(4 y z-y^{2}-8 z^{2}+z\right) a+\left(y z-6 z^{2}\right)\right]}\left[\begin{array}{c}
\left.\left[(y-4 z) a^{2}+\left(16 y z-4 y^{2}+y-5 z\right) a+4 y z-z\right)\right]
\end{array}\right.
$$

and the explicit evaluation of the symmetric functions leads to

$$
\begin{align*}
& z^{4}\left[-6144 y^{2}+2816 y-320\right]+z^{3}\left[4096 y^{3}-2400 y^{2}+444 y-25\right]  \tag{17}\\
& +z^{2}\left[-896 y^{4}+608 y^{3}-136 y^{2}+10 y\right]+z\left[64 y^{2}-48 y^{4}+12 y^{3}-y^{2}\right]=0
\end{align*}
$$

which reduces to

$$
\begin{equation*}
T \equiv z(4 y-1)[4 y(y-6 z)-y-5 z)]\left[4(y-4 z)^{2}-(y-5 z)\right]=0 \tag{I8}
\end{equation*}
$$

For convenience we call

$$
\begin{aligned}
& 4 y(y-6 z)-(y-5 z): T_{1}, \\
& 4(y-4 z)^{2}-(y-5 z): T_{2} .
\end{aligned}
$$

As in the internal case the vanishing of the denominators must also be discussed. The condition that the numerator and denominator for $a$ should both vanish is

$$
\begin{equation*}
T_{2}\left[z^{2}(32 y-z)+z\left(128 y^{3}-64 y^{2}-y+z\right)+\left(-3^{4}+20 y^{3}-3 y^{2}\right)\right]=0 \tag{19}
\end{equation*}
$$

The second factor is however an extraneity as it does not also consist with $a^{3}-a^{2}+a y-z=0$.
$T_{2}$ may simply be expressed by $k=-4$ or as $a(a-1)+4 z=0$.
The latter relation multiplied by the corresponding expressions in $b, c$ and expressed in $y, z$ is $z^{2} T_{2}=0$, while if $k$ be replaced by -4 in the equation for $a$ ( I 2 ) this becomes the square of $a(a-1)+4 z=0$.
$T_{2}=0$ is not however a proper discriminantal factor, for the value $k=-4$ has no relation to the homogeneous problem, and if $a+b+c \neq \mathrm{I}$ the $a$ equation does not become a square. Moreover the expression for $\delta k$ becomes a perfect square for $k=-4$ and we do not obtain two distinct solutions in the oneighborhood.
$T_{z}$ in the $a, \beta$ plane is represented by $\beta(4-\alpha)-8 a=0$, in which the factor $P^{2}$ enters so that the triangles given along $T_{2}=0$ are in a sense associated with isosceles triangles. None are real or possible.

The factor $T_{\mathrm{t}}$ is purely extraneous. If $T_{\mathrm{r}}=0, z=\frac{\left(4 y^{2}-y\right)}{(24 y-5)}$, which is satisfied by $a=\frac{1}{2}$, $b=\frac{1}{4}, c=\frac{1}{4}: y=\frac{5}{16}, z=\frac{1}{3^{\frac{1}{2}}}$.

For these values

$$
\beta=\infty, u=4 \text { but } \beta(\alpha-4)=\infty .
$$

Namely for $T_{1}=0$

$$
\begin{gathered}
u-4=-\frac{4(16 y-5)}{64 y^{2}-5} \\
\beta=\frac{\mathrm{I} 28(3 y-1)(24 y-5) y}{(16 y-5)^{2}} .
\end{gathered}
$$

For this set of values the equation for $\sigma$ has roots $0,0,-1,-1,-\frac{1}{4}$, and $-\frac{2}{3}$ and the last named, a single root, is the value giving the triangle.

Taking up the factor $z$ we have for $z=0, B=\infty, A=-\frac{(3 y-1)}{(4 y-1)}$ if $y \neq 0, y \neq \frac{1}{4}$. For $B=\infty$ the equation $F(y, A, B)=0$ becomes $(4 y-1)^{2} y\left[16 y^{3}(4 A+3)^{2}-8 y^{2}\left(32 A^{2}+52 A+2 \mathrm{I}\right)+y\left(84 A^{2}+\right.\right.$ $\left.\left.{ }_{147} A+64\right)-(9 A+8)(A+\mathrm{r})\right]=0$ and $y=\frac{(A+\mathrm{r})}{(4 A+3)}$ is a single root. Hence $z=0$ is not discriminantal unless $y=\frac{1}{4}$ or $y=0$ and these points occur on the other factors. The factor $z$ is then extraneous.

We are left with ( $4 y-1$ ) which is in fact a discriminantal factor and will be referred to as $\bar{T}$. In the $A, B$ plane it is represented by ${ }_{27} A-B=0$ and in the $a, \beta$ plane by $(\alpha-4)(\beta+27)+8 \mathrm{I}=0$.

Using a similar notation to the internal case we write $\bar{D}_{2}=0$ as the representative of the equal-bisector non-isosceles locus.

If $k=l, a \neq b$, there are in the $(a, b, c)$ plane three factors of the form

$$
\bar{D}_{2}(a, b) \equiv 2(a+b)^{3}+4 a b(a+b)-5(a+b)^{2}-3 a b+4(a+b)-1=0 .
$$

Expressing the $(a, b)$ form in $c$ and $z=a b c$ if $a+b+c=1$ we have

$$
2 c^{4}-c^{3}+4 c z-z=0 .
$$

Reducing by $c^{3}-c^{2}+y c-z=0$, we have $c=0$ or

$$
c=-\frac{y-6 z}{2 y-1} .
$$

Eliminating $c$

$$
\begin{equation*}
\bar{D}_{2}(y, z) \equiv 16 y^{3} z-4 y^{4}+216 z^{3}-180 y z^{2}+30 y^{2} z+y^{3}+36 z^{2}-12 y z+z=0 \tag{20}
\end{equation*}
$$

This factor of the discriminant has of course the same representation in the $a, \beta$ plane as in the internal case.

For $\bar{D}_{2}=0$ the sextic becomes a perfect square. If $k=l=\mathrm{I}$,

$$
B=m^{2}-2 m-8, \quad A=-\frac{\left(m^{2}+2 m+3\right)}{(m+2)^{3}}
$$

With these values $F(y, A, B)$ becomes

$$
\begin{equation*}
\left[16 y^{3}(m-4)^{2} m-8 y^{2}(m-4)\left(m^{2}-8 m-2\right)+y\left(m^{3}-21 m^{2}+75 m+53\right)+\left(m^{2}-8 m-11\right)\right]^{2} \tag{2I}
\end{equation*}
$$

In the reduction the factor $(m+2)^{4}$ is removed from both numerator and denominator. For $m=-2, a=0, \beta=0$ and the affair is indeterminate. For this case however $A=\infty, B=0$ with $A B=\infty, A B^{2}$ finite, and the limiting values serve.

We conclude from the set of three pairs of equal roots that as in the internal case $\bar{D}_{2}^{3}$ is a probable factor of the discriminant.

An expression for the discriminant in the $(y, z)$ plane may be obtained by equating the values of the derivatives $\frac{d y}{d z}$ as given by the two equations (8) ( $8^{\prime}$ ) and eliminating $A, B$. This process gives

$$
D(y, z) \equiv(4 y-1)\left(4 y^{2}-y-3 z\right)^{3}\left[D_{2}(y, z)\right]=0 .
$$

Hence as $4 y^{2}-y-3 z$ vanishes only for $\alpha=0, \beta=0$ no new factors are obtained by this method.

## 6. THE NODAL CURVE

Among the factors of the discriminant of $F(y, A, B)$ is one which relates to equality of the $y$ 's only without implying equality of the $z$ 's and hence not discriminantal for the problem. This arises from the nature of the elimination process: the two quadratics from which $z$ was eliminated may become identical. There are two conditions in $y, A, B$ from which $y$ can be eliminated. The result is

$$
A B(a-4) P=0
$$

where $P$ expressed in $\alpha, \beta$ is

$$
\begin{equation*}
P \equiv \beta^{2}(\alpha-4)(\alpha-9)+54 \beta(\alpha-1)(2 \alpha-9)+7^{2} 9^{\alpha}(\alpha-\mathrm{I}) \tag{22}
\end{equation*}
$$

When $P=0, y$ is given by

$$
\begin{equation*}
y=\left[\beta(\alpha-4)(\alpha-9)-27^{\alpha}(a-1)\right]+\left[\mathbf{1}_{2} \beta(\alpha-4)(a-3)\right] \tag{23}
\end{equation*}
$$

Since $\alpha=\infty$ does not cause $A$ or $B$ to vanish the effective factors are $(1-\alpha)(\alpha-4)^{2} P$.
The line $\alpha=1$ is in fact a nodal line on the surface $F(y, A, B)=0 . \quad P=0$ is also a nodal line while $a-4=0$ gives an infinite cuspidal edge.

The last locus is discriminantal for the problem while the others are merely so in relation to the choice of $y$ and the elimination method.

## 7. FINITE MULTIPLE POINTS OF ORDER HIGIIER TIIAN TIIE SIECOND

If $D_{2}=0, \bar{T} \neq 0$ and if no other possible factor of the discriminant vanishes the only possibilities are 4 -points and 6-points.

The discriminant of the cubic factor to the square of which $F(y, A, B)$ reduces for $D_{2}=0$ is to a numerical factor

$$
(m-4)^{3}(m-1)^{6}\left(m^{2}+m+7\right) .
$$

There should also be counted $m=\infty \quad(a=\infty, \beta=\infty)$.
For $m=1: \alpha=3, \beta=27$ occurs the 6-point $y=\frac{1}{2}$ already discussed.
For $m=4: a=4, \beta=$ any finite value, four values of $y$ are infinite.
The only real finite $\beta$ on $D_{2}=0$ and on $a=4$ is 54 .
The complex factors are intimately connected with $n=4$ in that if $n=\frac{(m-1)}{3}$, $(m-4)\left(m^{2}+m+7\right)=27\left(n^{3}-1\right)$. They give

$$
\begin{array}{ll}
a=\frac{3 \pm \sqrt{-3}}{2} & \beta={ }_{14}^{9}(-14 \pm \sqrt{-3}) \\
A=\frac{-3 \mp \sqrt{-3}}{6} & B=-\frac{9}{2}(3 \pm 1 / \overline{-3})
\end{array}
$$

and are intersections of $\bar{T}$ and $D$.
For $\bar{T}=0$ we have $y=\frac{1}{4}$ which is a double root for all $A$ 's when $27 A=B$ and a fourfold root when $3 A^{2}+3 A+1=0$. These values give the complex 4 -points just mentioned and there is never a 6 -point.

The list of multiple points is then

| $\begin{aligned} & a=3, \beta=27 \\ & a=\frac{(3 \pm \sqrt{-3})}{2} \end{aligned}$ | a 6-point two 4-points | on $D_{2}, P$, not on $\bar{T}$ on $D_{2}$ and $\bar{T}$ |
| :---: | :---: | :---: |
| $\alpha=4, \beta$ any | a 4-point line |  |
| $\alpha=0, \beta=0$ | is indeterminate |  |
| If $L \frac{\beta}{\alpha} \neq 0 \neq \infty$ | a 3-point | $y=\frac{1}{4}$ on $\bar{T}$ |
| If $L \frac{\beta}{u^{2}}=k$ | a 4-point | $y=\infty$ |
| $\alpha=\infty, \beta=\infty$ | a double 3-point | $y=0, y=\frac{1}{4}$ |
| $\alpha=\infty, \beta=0$ | a 4-point | $y=\infty$ |
| $\alpha=0, \beta=\infty$ | is indeterminate | (a 3-point if $B$ is finite) |

## 8. THE DISCRIMINANT

The following factors have been found:

$$
\begin{gathered}
D_{2}(A, B) \equiv B^{2}(A+\mathrm{r})^{2}+2 B(9 A+7)+27(4 A+3) \\
P(A, B) \equiv B^{2}(9 A+8)-54 A B(9 A+7)-729 A(4 A+3) \\
\bar{T}(A, B) \equiv 27 A-B
\end{gathered}
$$

$A ; B ; 4 A+3 ;$ and $B=\infty$ is also discriminantal.

If any other discriminantal loci exist they must have common points with either $A=0$ or $B=\infty$.

For $A=0 B \neq 0$ the equation becomes

$$
B(4 y-1)^{2}(2 y-1) y-(3 y-1)(6 y-1)=0
$$

after dividing out the pair of factors $(3 y-1)^{2}$ which belong to $A=0$. Further equalities occur for $B=0, B=\infty$ and for values of $y$ which are roots of

$$
288 y^{4}-288 y^{3}+104 y^{2}-16 y+1=0
$$

the corresponding value of $B$ being given by

$$
B\left(32 y^{2}-16 y+1\right)=9
$$

The intersections of $A=0$ and the locus $D_{2}$ give for $B$ the quadratic

$$
B^{2}+14 B+8 \mathrm{I}=0
$$

In the field of the complex roots of this equation the quartic for $y$ reduces and gives the set of four $y$ 's, two for each value of $B$, and as $D$ is the locus of three pairs of equal roots we have no outstanding discriminantal points on $A=0$.

For $B=\infty$ the equation becomes

$$
A^{2}(4 y-1)(8 y-3)^{2}+A(3 y-1)\left(128 y^{2}-96 y+17\right)+8(3 y-1)^{2}(2 y-1)=0,
$$

after the pair of factors belonging to $B=\infty\left[(4 y-1)^{2}\right]$ and the factor $y$ have been set aside. The values of $A$ for which $y$ is a double factor are -1 , and $-\frac{8}{9}$. The former belongs to $D_{z}$ and the latter to $P$, and the system of equalities is in each case what is required of such intersections. The discriminant of the cubic in $A$ is

$$
64 A^{2}(4 A+3)^{4}(A+1)
$$

and since the term $A^{8}$ is absent $A=\infty$ must be discussed.
For $A=0$ one pair of equal roots occurs in keeping with the discriminantal character of $A$. $A=-\mathrm{I}$ belongs to $D_{2}$ and has in all four roots $-\frac{1}{4}$ and two $y=0$. This is in accord with the fact that $D_{a}$ has normally 3 pairs and $B=\infty$ is simply discriminantal. $A=-\frac{4}{3}$ is itself a factor and the two pairs occurring are expected. $A=\infty$ has no extra equalities for $B=\infty$.

We conclude that the factors occurring in the discriminant are $A, 4 A+3, B, \bar{T}, P, D_{2}$; that $A=\infty$ and $B=\infty$ are discriminantal and that there are no others.

The complete discriminant is then of the form

$$
N \cdot A^{a} \cdot B^{b} \cdot(4 A+3)^{m} \cdot \overline{T^{b}} \cdot P^{p} \cdot D^{d}{ }_{2}
$$

To determine the exponents and the numerical factor $N$ special values of Bezout's determinant are calculated.

First put $A=-1$, a non-discriminantal value, and $B=27$. The computation is rendered comparatively light by transforming $y$ to $\frac{(\eta+1)}{4}$ and multiplying up by 16 to obtain an integral form. This divides the discriminant by $2^{20}$. Writing $B=27 x$ and dividing by 27 we have the form

$$
{ }_{27} x^{2} \eta^{4}(\eta+1)^{2}+x(3 \eta-1)\left(9 \eta^{3}+9 \eta^{2}+3 \eta-1\right)+(3 \eta-1)^{2} .
$$

The value of the Bezout determinant for this form is if $x=1$

$$
2^{15} \cdot 3^{24} \cdot 5^{3} \text { and if } x=2,2^{20} \cdot 3^{29} \cdot 13^{2}
$$

The algebraic factor of $T$ is $x+1$, of $P,\left(x^{2}+4 x+1\right)$, and of $D_{2},(4 x+1)$. From these values we conclude that $d=3, p=2, t=\mathrm{r}, b=8$. The values of $a$ and $m$ cannot be cletermined since both $A$ and $(4 A+3)$ are $(-1)$.

Replacing the factors divided out in the transformation we have

$$
\Delta= \pm 2^{32} \cdot 3^{52} \cdot A^{a} \cdot B^{8} \cdot T \cdot P^{2} \cdot D_{2}^{3}(4 A+3)^{m}
$$

To determine $a$ and $m$ we give $A$ and $B$ simple non-discriminantal values and calculate the residue of the Bezout determinant modulo a suitable number prime to all the factors of $\Delta$.

$$
\begin{aligned}
& A=\mathrm{I}, B=\mathrm{I} \text { gives } 4 \cdot 7^{m} \equiv-2(\bmod \mathrm{II}) \\
& A=2, B=\mathrm{I} \text { gives } 2^{a+\mathrm{r}} \equiv-2(\bmod 5) \\
& A=2, B=\mathrm{I} \text { gives } 2^{a+2 m} \equiv \mathrm{I}(\bmod 7)
\end{aligned}
$$

That is provided the positive sign is taken with $\Delta$. The only solutions permissible on account of the limitations of the order are $a=2, m=2$. There is no permissible solution with the negative sign for $\Delta$.

On account of the connection in general between the Bezout form and the standard form of the discriminant the result is to be divided by $-6^{4}$ and we have finally

$$
-2^{28} \cdot 3^{4^{8}} \cdot A^{2} \cdot B^{8} \cdot \bar{T} \cdot P^{2} \cdot D_{2}^{3}(4 A+3)^{2}
$$

as the value of the discriminant.

## 9. the interrelations of the two equations

The variables $y$ and $\sigma$ are connected by a birational transformation, namely

$$
\begin{equation*}
y=\frac{A+\mathrm{I}+\sigma}{4 A+3-3 \sigma^{2}} \quad \sigma=\frac{8 A B\left(4 y^{2}-y\right)+8 B(3 y-\mathrm{r})+9 A}{B\left[3 A(4 y-1)(8 y-3)+\left(3 y^{\mathrm{I}}-\mathrm{I}\right)^{2}\right]} \cdot \frac{\mathrm{I}-3 y}{y} \tag{24}
\end{equation*}
$$

In general the discriminantal factors will be identical but at certain critical points and lines irregularities may occur.

For $A=-\frac{2}{3}(a=3)$ the equation in $\sigma$ bas three roots equal to $-\frac{1}{3}$ and the expressions for $y$ become indeterminate. The proper values for $y$ are in this case the three roots of the expression for $\sigma$ considered as a cubic in $y$. In fact the locus $A=-\frac{2}{3}$ is not discriminantal for the $y$ equation but a locus of reducibility, the reduced factors being

$$
\begin{equation*}
B(4 y-1)^{2}(y-1)-9(3 y-1) \text { and } B\left(8 y^{3}-12 y^{2}+3 y\right)-9(3 y-1) \tag{25}
\end{equation*}
$$

In a similar way for $A=0$ the $\sigma$ equation has four roots equal to -1 and the expression for $y$ is indeterminate, and that for $\sigma$ fails to give the $y$ values which must be sought from the $y$ equation which has distinct roots for these four places.

Similar irregularities occur for $B=0, B=\infty$, and $A=\infty$. They are complicated by the fact that the connection between $(A, B)$ and $(\alpha, \beta)$ is also birational. These do not call, however, for any special computations.

## IO. THE TRANSFORMATIONS*

To avoid the birational transformation from $a, \beta$ to $A, B$ and to keep as close to the triangles as possible we consider the transformations leading in a chain from the sides $(a, b, c)$ to the symmetric functions of the sides $(x=1, y, z)$ and to the symmetric functions ( $\alpha, \beta$ ) of the angle-bisectors which may be taken as the data of the problem. The equation $F(y, A, B)=0$ will then be considered as if its coefficients were explicitly written in $(\alpha, \beta)$.

We trace the discriminantal loci in the ( $a, b, c$ ) plane. Beginning with $\alpha=\infty$, the representative is

$$
\Sigma a b(b-c)^{2}(c-a)^{2}(a-b+c)(-a+b+c)=0 .
$$



This is a curve of the eighth order with sixfold symmetry. It is not difficult to establish the following features. The curve has no real infinite branches. The center of the triangle of reference is a conjugate point. The curve has two branches at each vertex touching the sides. It has two branches at the mid-points of the sides touching with inflexion the lines $a+b-c=0$, etc. The extent to which the branches leave the sides is determined sufficiently by the points,

$$
\begin{array}{lll}
a=.205 & b=.045 & c=.75 \\
a=.465 & b=-.215 & c=.75 \\
a=.06 & b=-.26 & c=1.2 \\
a=.2 \mathrm{I} & b=-.4 \mathrm{I} & c=\mathrm{I} .2
\end{array}
$$

and the symmetric correspondents (Fig. 17).
The locus $\beta=\infty$ has three representatives: $(a-b)(b-c)(c-a) ; a b c$, and $(a+b-c)(a-b+c)$ $(-a+b+c)$.

The loci $\alpha=0$ and $\beta=0$ are jointly represented by

$$
\mathbf{\Sigma} a(b-c)^{2}(b+c-a)=0 .
$$

This curve has sixfold symmetry and is closed. It may be traced by writing $a=x+y, b=x-y$, $c=1-2 . x$ when it takes the form

$$
2 y^{4}+y^{2}\left(12 x^{2}-11 x+2\right)+\left(18 x^{4}-21 x^{3}+8 x^{2}-x\right)=0 .
$$

As a quadratic in $y^{2}$ the discriminant is

$$
-96 x^{3}+105 x^{2}-36 x+4
$$

whose one real zero $x=.549$. . . limits the curve to a triangle slightly larger than the reference triangle.

The curve passes through the vertices parallel to the opposite sides and meets the sides also at the midpoints touching them there. It has no singularities except a conjugate point at the center of the reference triangle.

The discriminantal factor $\bar{T}$ is represented by $4(a b+b c+c a)-1=0$. This is the inscribed circle of the reference triangle. The factor $D_{2}$ is represented by three symmetrical curves of which $D_{2}(a, b)$ is

$$
\begin{equation*}
2(a+b)^{3}+4 a b(a+b)-5(a+b)^{2}-3 a b+4(a+b)-1=0 \tag{26}
\end{equation*}
$$

Writing $a+b=2 x, a-b=2 y$ we have

$$
y^{2}=\frac{\left(8 x^{2}-5 x+1\right)(3 x-1)}{8 x-3}
$$

$y$ is real except for $\frac{8}{24}<x<\frac{9}{24}$.
The asymptotes are

$$
y=\sqrt{3} x-\frac{\sigma^{7}}{2} \sqrt{3} \text { and the conjugate. }
$$

The intersections with the asymptotes are at $x={ }_{13}^{4} \mathbf{3}^{3}$.
The curve passes through the center of the triangle, the midpoints and the vertices where it touches the sides. At the midpoints the direction is $\frac{\delta a}{\delta b}=-\frac{\delta}{6}$ (Fig. I8).

By determining these curves and their intersections the diagram of a one-sixth part of the $(a, b, c)$ plane symmetrical and serving as a fundamental region is obtained. It has sixteen compartments (Fig. 19):

Regions (1), (2) corresponding to real possible triangles with three external bisectors, regions (3), (4), (5), (6) , (7) , (8), (11), (I2), (13), (14), corresponding to real impossible triangles with pure imaginary bisectors, regions (9), (10) with real possible triangles, one bisector being external and two internal, and regions ( 15 ), ( 16 ) with impossible triangles and real bisectors.

After the symmetric function transformation these compartments are to be followed to the $(y, z)$ plane.

$$
\text { The } y, z \text { Plane }
$$

The locus $a=\infty$ is represented by

$$
4 y^{4}-y^{3}+9 z^{2}-6 y^{2} z-3 y z+z=0
$$

Real points only occur in general for $|y|<\frac{1}{6} / \overline{3}$. There is however a conjugate point at $y=\frac{1}{3}, z=\frac{1}{2}$..$~ z$ has a maximum for $y=\frac{(1+\sqrt{13})}{16}, z=\frac{\left(131^{\prime} 13-35\right)}{576}$, and a minimum at the conjugate. At the origin there is an inflexion $y^{3}=z$.

The sides and bisectors of the reference triangle transform as in case of the internal problem, the real region being inside the curve $D_{1}$ (Fig. 20). The closed curve $a=\infty$ lies entirely inside the curve $D_{1}$.
$\alpha=0, \beta=0$ is represented by the parabola $3 z-4 y^{2}+y=0$. This passes through the origin 0 , the point $F\left(\frac{1}{4}, 0\right)$, the point $A\left(\frac{1}{3}, \frac{1}{27}\right)$ and cuts the line $4 y-8 z-\mathrm{I}=0[\beta=\infty]$ at $F$ and $Q\left(\frac{3}{8}, 1_{6}^{1}\right)$ and has no contacts with the other curves in the real region $(y, z)$.


Along $D_{1}(\beta=\infty, a=4)$ is represented. $z=0$ represents $\beta=\infty$ and so does $4 y-8 z-1=0$. The curve $D_{2}$ is

$$
\begin{equation*}
16 y^{3} z-4 y^{4}+216 z^{3}-180 y z^{2}+30 y^{2} z+y^{3}+36 z^{2}-12 y z+z=0 \tag{27}
\end{equation*}
$$

This approximates the semicubical parabola $27 z^{2}+2 y^{3}=0$ in the infinite regions, and has an asymptote $32 y-128 z-5=0$, the further intersections with which are complex. There is a conjugate point at $\left(\frac{1}{2}, \frac{1}{2}\right)$ on the ordinary branch, an inflexion at the origin $y^{3}+z=0$, and no
other singularity. It touches $D_{1}$ at $\left(\frac{1}{4}, 0\right)$ having $4 y-8 z-\mathrm{I}=0$ as the tangent and also touches $D_{\mathrm{I}}$ at the cusp $A$. From the cusp out to $y=\infty \quad D_{\mathrm{I}}$ is outside $D_{2}$, otherwise inside.
$\bar{T}$ is represented by the line $y=\frac{1}{4}$.
There is no difficulty in identifying the regions ( 1 ) to ( 16 ) in the ( $y, z$ ) diagram. They

cover the interior of $D_{1}$. The exterior is divided by the curves into twelve regions which are marked I, II, . XII as a basis for discussing the transformation to the ( $\alpha, \beta$ ) plane.

$$
\text { The a, } \beta \text { Plane, Limits }
$$

Since the transformation from $(y, z)$ to $(\alpha, \beta)$ is not everywhere point for point it is necessary to investigate certain limits.


Limits for infinite values of $y, z$ are:
Along $z=m y^{2}$

$$
a=\frac{(3 m-4)^{2}}{9 m^{2}-6 m+4} \quad \beta=0 .
$$

For $m=\infty: a=1, \beta=0$
for $m=0: a=4$ but $\beta=0$ is not reached, $z=0$ giving $\beta=\infty$. The point $\alpha=4, \beta=0$ is attained as nearly as we please in VIII, IX and also in (15), (14). Any positive $\alpha<4, \beta=0$ is reached in IV or V and IX and also in I, XI, the parabolas being eventually outside $D_{\mathbf{r}}$.

Along $z^{2}=m y^{3}$

$$
a=4, \beta=\frac{-8}{m(27 m+4)},
$$

$\alpha=4, \beta$ any negative is reached in I, XI, V, IX. At $m=-\frac{4}{2} 7$ the curve eventually falls on $D_{\mathbf{r}}$ but $\beta=-\infty$ so that no limitation is set on $\beta$ in XI.

For the limits at $A\left(\frac{1}{3}, 2^{\frac{1}{7}}\right)$ writing $y=\frac{1}{3}+\lambda, z=\frac{1}{2} \frac{1}{7}+\rho$ we have

$$
\alpha=\frac{(5 \lambda-9 \rho)^{2}}{13 \lambda^{2}-63 \lambda \rho+81 \rho^{2}} \quad \beta=\frac{27(5 \lambda-9 \rho)^{3}}{3(\lambda-3 \rho)^{2}+4^{\lambda^{3}}} .
$$

Using the tangents to the $D_{\Sigma}$ curve and the parabola representing $a=0, \beta=0$ as axes

$$
\begin{array}{r}
5 \lambda-9 \rho=X \\
\lambda-3 \rho=Y
\end{array}
$$

and approximating along semicubical parabolas $Y^{2}=m X^{3}$, we obtain

$$
a=4, \beta=\frac{54}{(6 m+1)}
$$

For positive $m$ 's $0<\beta<54$. For $0>m>-\frac{1}{6} \beta$ is positive and the curve lies in regions (I), (2). For $m<-\frac{1}{6}$ the curve is in VII, XII.

Hence VI, VIII cover $a=4$ from $\beta=0$ to 54,

$$
\text { VII, XII cover } a=4 \text { from } \beta=0 \text { to }-\infty \text {, }
$$

(1), (2) cover $a=4$ from $\beta=54$ to $+\infty$.

At $A$ if $\lambda=p \rho, a=\frac{(5-9 p)^{2}}{13-63 p+81 p^{2}}$ and $\beta=0$.
For $p=3, a=4$ and for $p=\frac{5}{9}, a=0$.
Hence values of $a$ between 0 and 4 are found paired in VII and VIII and also in XII and VI.
Limits at $F\left(\frac{1}{4}, 0\right)$ are investigated by writing $y=\frac{1}{4}+\lambda, z=\rho$.
We have

$$
a=\frac{16(3 \rho-\lambda)^{2}}{\lambda-2 \rho} \quad \beta=-\frac{(3 \rho-\lambda)^{3}}{\rho(\lambda-2 \rho)^{2}}
$$

Using $(\lambda-2 \rho)$ the tangent to $a=\infty$ and $\beta=\infty$ as $X^{\circ}$ axis and $\lambda-3 \rho$ the tangent to $a=0, \beta=0$ as $Y$ axis.

$$
a=\frac{16 X^{2}}{Y} \quad \beta=\frac{X^{3}}{Y^{2}(Y-X)}
$$

For $Y=m X^{2}: \alpha=\frac{16}{m}, \beta=\infty$, for $Y=m X: a=0, \beta=\frac{1}{m^{2}(m-1)}$.

$$
\text { For } Y=m X, m=\left(2-\frac{\lambda}{\rho}\right) \div\left(3-\frac{\lambda}{\rho}\right) \text {; }
$$

and we have
$m>1: \beta>0$ the region VIII, $\frac{\lambda}{\rho}>3$
$\frac{2}{3}<m<1: \beta<0$ the region IX, $\frac{\lambda}{\rho}<0$,
$0<m<\frac{2}{3}: \beta<0$ the region X, $0<\frac{\lambda}{\rho}<2$.

For $\frac{\lambda}{\rho}$ between 2 and $3, m$ is negative and the regions are the real sheets ( 1 ), (2), (3), (5), (6), (7), (I2), (II), (10), (9).

Limits at $Q\left(\frac{3}{8}, 1_{6}^{16}\right)$.
Here $4 y-8 z-\mathrm{I}=0,(\beta=0)$ and $3 z-4 y^{2}+y=0,(\alpha=0, \beta=0)$ cut.

$$
\text { If } y=\frac{3}{8}+\lambda, z=\frac{1}{16}+\rho: \alpha=0, \beta=16(2 \lambda-3 \rho)^{3} \div 217(\lambda-2 \rho),
$$

and cubical parabolas give $\alpha=0, \beta$ any.
$\beta$ is positive in V, VII and negative in IV, VI.
Limits at $O(\mathrm{o}, \mathrm{o})$.

$$
a=(y+3 z)^{2} \div\left(z-y^{3}\right), \beta=(y+3 z)^{3} \div\left(4 z-y^{2}\right) z
$$

are proper approximations.
Along $z=m y^{2}: \alpha=\frac{I}{m}, \beta=\infty$.
These parabolas fall in III for $m>\frac{1}{4}: \beta>0, \alpha<4$ and do not fall in II but in I, which is limited by $a=4$. Hence I and III are joined at $\beta=\infty, 0<\alpha<4$.

Along $y^{3}=4 \beta z^{2}: a=0$. These curves all fall outside $D_{1}$, and join I to ( 13 ) for $\alpha=\infty$ at $O$ is $y^{3}+z=0$ and is closer to the axis than all the semicubical parabolas.

The point $J\left(\frac{1}{2}, \frac{1}{2}\right)$ is a conjugate point on $D_{2}$ and $y=\frac{1}{2}$ is a sixfold root for $\alpha=3, \beta=27$. Two sheets from the real $y, z$ regions are VI and VIII: the other four sheets are complex in the neighborhood.

In transforming to the $(\alpha, \beta)$ plane we notice first that for the whole of the regions outside $D_{1}, 0<\alpha<4$.

For $a-4=\frac{P^{2}}{D_{a}}$ where $P^{2}$ is the discriminant of the equation whose roots are the sides and is negative, while $D_{a}$ the denominator of $\alpha$ only vanishes on a closed curve inside $D_{\mathrm{r}}$ and is positive outside: while $a=\left(3 z-4 y^{2}+y\right)^{2} \div D_{a}$ is positive (3) (II, I).

$$
\text { The a, } \beta \text { Plane }
$$

In this plane we trace:

$$
\begin{align*}
& D(\alpha, \beta) \text { as in case I. } \\
& \bar{T}(\alpha, \beta) \equiv(\alpha-4)(\beta+27)+8 \mathrm{I}=0  \tag{28}\\
& P(\alpha, \beta) \equiv \beta^{2}(\alpha-4)(\alpha-9)+54 \beta(\alpha-1)(2 \alpha-9)+729 \alpha(\alpha-1)=0  \tag{29}\\
& \alpha=0, \alpha=\mathrm{I}, \alpha=4, \beta=0 .
\end{align*}
$$

These are all the finite discriminantal lines.
The locus $P=0$ has no representative in the $y, z$ plane but effects there a pairing of points which come together in the $\alpha, \beta$ plane forming a nodal line in the $y, \alpha, \beta$ surface. The locus

$a=\mathrm{I}$ is of the same character while $\alpha=4$ and $\beta=0$, which are infinite cuspidal edges on the surface, partake of this property.

The regions (r), (2) being within $D_{\mathrm{I}}$ and outside $\alpha=\infty$ in the $y, z$ plane have $a>_{4} \beta \geq 54$ and reach $\alpha=4$ at $A$ for any $\beta \geq 54$, and $a=\infty, \beta=\infty$ at $F$. Being divided by $D_{2}$ they fold on $D$ and on $\alpha=4$ and cover the region inside $D$ 's cusp and above $a=4$.

The regions (7), (8) are within $D_{\mathrm{r}}$ and have $\alpha<0, \beta>0$ and join on $D_{2}$.
In $(\alpha, \beta)$ they fold on $D$ in the fourth quadrant reaching $a=0, \beta=0$ and $a=-\infty, \beta=\infty$.
The regions (3), (4) are separated by $\bar{T}$ in $(y, z)$, are within $D_{\mathrm{I}}$ and outside $\alpha=\infty$. Hence $\alpha>4, \beta<0$. They reach $\alpha=4, \beta=\infty$ on $\bar{T}$ and $D$, also $\alpha=\infty, \beta=-27$ on $\bar{T}$ and $\alpha=\infty\left(y=\frac{1}{4}\right.$, $z=\frac{1}{7}$ ).

The regions (5), (6) also double on $\bar{T}$ but here $a<0, \beta<0$.
The regions (I5), (I6) have $a \geq 4, \beta>0$ and fold on $D$. They reach $a=4, \beta=54$ along $D_{2}$ at infinity. $D_{2}$ is here approximately $27 z^{2}+2 y^{3}$. (15) reaches $a=4, \beta=\infty_{0}$ along $z=0$, and (r6) attains the same values along $D_{1}$. The values $a=4,54<\beta<\infty$ are reached for various $m$ 's along $z^{2}=m y^{3}$ whose limits are $\alpha=4, \beta=-\frac{8}{m(27 m+4)}$. The $D_{2}$ value $m=-\frac{2}{2} 7$ being a turning-point for $\beta$ 's denominator.
(9), (IO) behave in a similar manner and also fall on the upper cusp region in ( $a, \beta$ ).
(ri), (14) are divided by $D_{2}$, reach $\alpha=4 \beta=\infty$ but $\beta<0$ throughout. They fold on $D$ 's hyperbolic branch in the second quadrant.
(I2), (I3) similarly fold on $D$ in the third quadrant. They reach $\alpha=\infty, \beta=\infty$ at $F$ and $O$ respectively and so fall on the negative side of $D$. They reach $\alpha=0, \beta=0$ with (7), (8).

Of the sheets I, II, . . XII with real $(y, z)$ but complex $(a, b, c)$ which all lie between $a=0$ and $a=4$ :

I reaches $a=4, \beta<0$ along $z^{2}=m y^{3}$ for $m<-\frac{4}{27}$, reaches $\beta=\infty, 0<\alpha<4$ at $O$ along $z=m y^{2}$ for $m>\frac{1}{4}$, reaches $\alpha=0,0>\beta>-\infty$ at 0 along $y^{3}=4 \beta z^{2}$, reaches $\beta=0,0<a<4$ along $z=m y^{2}$ for $z=\infty, 0<m<\frac{4}{3}$, and thus covers the rectangle $0<\alpha<4, \beta<0$.

II and III are continuous over $y=0$ which is not discriminantal. They reach $a=0$, $0<\beta<\infty$ at o along $y^{3}=4 \beta z^{2}$. The values are here paired with those in (8) since $D_{2}$ is closer to the axis than any of the parabolas. The sheet reaches $\beta=0,0<a<1$ for $z=m y^{2}, z=\infty$. It does not reach $a>\mathrm{I}$ but is folded with IV along $\bar{T}$.

IV reaches $\beta=\infty, 0<\alpha<4$ along $Q Z: \alpha=0,0<\beta<\infty$ at $Q$ on cubic parabolas; joins VII along $\beta=\infty, 0<\alpha<4$; reaches $\beta=0,0<\alpha<$ I for $z=m y^{2}$ pairing with II, III.

V covers the whole rectangle $\beta<0,0<\alpha<4$. It reaches $\alpha=0, \beta<0$ at $Q: a=4, \beta<0$ on semicubical parabolas at infinity: $0<\alpha<4, \beta=0$ for $z=m y^{2}$ at $z=\infty: 0<\alpha<4, \beta=\infty$ along $4 y-8 z-\mathrm{r}=0$ from $Q$ to $\infty$, a increasing monotonously.

VI reaches $\alpha=0, \beta>0$ at $Q$ the cubical parabolas pairing the values with IV; $\alpha=4$, $0<\beta<54$ at $A: 0<\alpha<4, \beta=\infty$ with V along $4 y-8 z-\mathrm{r}=0$. It contains $D_{2}$ and $J$ which goes to the cusp and is folded with VIII along $D$.

VII covers the rectangle $\beta<0,0<a<4$. It reaches $a=4,0>\beta>-\infty$ at $A: \alpha=0$, $0>\beta>-\infty$ at $Q: \beta=\infty, 0<\alpha<4$ along $Q Z$ joining IV; and $\beta=0,0<\alpha<4$ at $A$.

VIII covers the same region as VI folding symmetrically on $D^{2}$.
IX contains $\bar{T}$ and $\beta<0 . \quad A(\alpha=0, \beta=0)$ is not reached. It reaches $\alpha=0, \beta=-\frac{27}{4}$ on $\bar{T}$ at $F: a=4, \beta<0$ on $z^{2}-m y^{3}$ for $z=\infty, m>0 . \quad \beta=\infty, 0<\alpha<4$ is reached with VIII along $z=0$ from $F$ to $y=\infty$ : and $\beta=0, \mathrm{I}<a<4$ on $z=m y^{2}$ for $z=\infty, m<0$.

X and XI cover the same region as IX folding on $\bar{T}$.
XII covers the negative rectangle. It reaches $\alpha=0, \beta$ any, at $F$ joining (12): $\alpha=4, \beta$ any, with VII at $A: \beta=0,0<a<4$ with VI at $A$ and $\beta=\infty, 0<\alpha<4$ at $F$ for $y=m x^{2}, m>4$.

The nodal curve $P=0$ does not affect the reality of the roots but causes the sheets to cross. It has asymptotes $\alpha=4, \alpha=9, \beta=27(-2 \pm \sqrt{3})$. There are no real points for $\mathrm{I}<\alpha<3$, nor for $\frac{2}{2} 5<\beta<27$. At $a=3, \beta=27$ is a cusp which falls on the cusp of $D$ with coincident tangent, but the $P$ curve includes the $D$ curve up to $\alpha=4, \beta=54$ where they cross, and up to $\alpha=4, \beta=\infty$ where they have contact. The curve $P$ meets $\bar{T}$ only at $\alpha=\mathrm{I}, \beta=0$ and two complex points $\alpha=\frac{(14 \pm \sqrt{-2})}{22}, \beta=-\frac{27(9 \pm 1 \overline{-2})}{83}$, the positive signs corresponding. $P$ and $D$ meet at the cusp and at $\alpha=4, \beta=\infty$ and also at the origin where the tangent to $P$ is $3^{\alpha-2 \beta}=0$.

The curves have also two intersections for $\alpha=\infty, \beta=-\frac{2}{4}$.
Since $y$ is given as a rational function of $(\alpha, \beta)$ it is real for all real points, but the $z$ 's may form a complex pair. The condition is

$$
12(\mathrm{I}-\alpha)\left(4 y^{2}-y\right)+\alpha(6 y-1)^{2}<0
$$

This is satisfied for the branch which reaches the origin in the third quadrant and for the branch between $\bar{T}$ and $D$ in the second quadrant, and in the region round the cusp of $D$ up to the crossings.

To determine which compartments of the $y, z$ plane are paired by $P$ we consider first the region inside the cusp of $D$ with $a>4$. The sheets involved are (1), (2), (9), (10), (15), (16). At $\beta=\infty$ for positive approach the roots are $0 \cdot \frac{1}{4}, \frac{1}{4}, \frac{1}{(4-\alpha)}, \frac{(\alpha-6 \pm \sqrt{\alpha})}{4(\alpha-4)}$.

Taking $\alpha=6$ as a typical case where $P$ acts they are

$$
\text { o, } \frac{1}{4}, \frac{1}{4},-\frac{1}{2}, \pm \frac{\sqrt{6}}{8}
$$

Now ( I ), (2), (9) can all reach $y=\frac{1}{4}$ but (2) is the only region reaching $y>\frac{1}{4}, \beta=\infty$. Hence (2) has the root $\frac{\sqrt{6}}{8}$ and ( 1 ) $=$ (9) has $\frac{1}{3}$. (I6) reaches $y=0$ but ( 15 ) cannot. (IO) and (I5) have then the negative values. At the point $\alpha=6$ on $D=0$ we have $(\mathrm{I})=(2),(9)=(\mathrm{I} 0),(\mathrm{I} 6)=$ (15) and as only one crossing occurs between this value of $\beta$ and $\beta=\infty$ it must be (10), (I6) which cross.

The cross-section of the surface has the arrangement of the diagram (Fig. 22).
Passing across $\beta=\infty$ (4) and (II) are paired by $P$ for $\alpha>9$.
Next consider the region $\beta<0,0<\alpha<1$. Where $P$ crosses $\alpha=0$ three roots are equal to $\frac{1}{4}$. These are in X, IX, XII since VIII has $\beta>0$. $P$ can only pair IX and XII since the $y$ 's must be the same and $a=0$ for $P$ gives $y=\frac{1}{4}$.

The region $\beta>0,0<\alpha<$ I. $\quad Q$ approached on $4 y-8 z-1=0$ has $\beta=\infty, a=0$. For small $\alpha$ 's we have roots in IV, VI and $\beta>0$. IV can only reach $\beta=0$ by moving to infinity along $z=m y^{2}$, while VI must reach $A$. The $y, z$ diagram (Fig. 20) shows that this entails the crossing of the projections on the $y$-axis.

The branches of $P$ at the cusp join complex $z$ 's. Using the continuity of the real surface as a guide we see that the sheets (10), (16) crossing on $P$ for $\alpha>4$ represent two roots which become equal and infinite at $a=4$ and then complex, and so the nodal line is as usual continued as a


Fig $22 \alpha$ sections of $F(y, \alpha, \beta)=0$
real isolated nodal line whose $a, \beta$ projection is the curve $P$ and whose $y$ values run from $\infty$ to $\frac{1}{2}$ at the cusp and back to $\infty$ at the end of the asymptotic branch.

The small arch of $P$ between the origin and $\alpha=1, \beta=0$ pairs IV and VI. From the origin along the first vertical asymptotic branch (12) and (13) are paired; crossing $a=\infty$ the sheets are (II), (14) and the nodal line is isolated in both these parts.

From $\alpha=1, \beta=0$ moving to $\beta<0$ the paired sheets are IX, XII until $\alpha=0$ is crossed, when (6), (I2) take their places. Passing $\alpha=\infty$ (4), (II) are paired. For the last two parts the nodal line is ordinary.

In a similar way the equalities further marked in the diagram may be established. The result though a compendious collection of information is not, as remarked in case $I$, a complete and consistent statement of all the facts. In particular as special defects the representation of the points $\alpha=0, \beta=0: \alpha=1, \beta=0: \alpha=4, \beta=\infty$, for which the equation becomes indeterminate, and the surface has a line parallel to the $y$ axis, is omitted.

A set of diagrammatic cross-sections of the surface $F(y, a, \beta)=0$ is given in Fig. 22.

## 1I. THE DETERMINATION OF THE SIDES

The side of a triangle with given bisectors is a root of a sextic equation whose coefficients are rational and unsymmetric in the bisectors.

To determine these sides we write

$$
a-b=\lambda a, b-c=\kappa a
$$

whence

$$
\begin{equation*}
b=a(\mathrm{I}-\lambda), c=a(\mathrm{I}-\lambda-\kappa) \tag{30}
\end{equation*}
$$

The fundamental formulas for external bisectors then give

$$
\begin{gather*}
\overline{K^{2}} \div \bar{L}^{2}=p=\frac{(1-\lambda)(1-\kappa)(\lambda+\kappa)^{2}}{(1-2 \lambda-\kappa) \kappa^{2}}  \tag{31}\\
\overline{K^{2}} \div \bar{M}^{2}=q=\frac{(1-\lambda-\kappa)(1+\kappa) \lambda^{2}}{(1-2 \lambda-\kappa) \kappa^{2}}  \tag{32}\\
p(1-2 \rho \kappa-\kappa)=(1-\rho \kappa)(1-\kappa)(1+\rho)^{2}  \tag{33}\\
q(1-2 \rho \kappa-\kappa)=(1-\rho \kappa-\kappa)(1+\kappa) \rho^{2}
\end{gather*}
$$

and if $\lambda=\rho \kappa$

From these quadratics $\kappa$ can be climinated and we have

$$
\begin{equation*}
\rho^{3}(\rho+1)^{3}+p \rho^{4}(\rho+1)-q \rho(\rho+1)^{4}-p^{2} \rho^{3}+p q \rho(\rho+1)+q^{2}(\rho+1)^{3}=0, \tag{34}
\end{equation*}
$$

and $\kappa$ is given rationally in $\rho$ by

$$
\begin{equation*}
\kappa=\frac{p \rho-q(\rho+1)-\rho(\rho+1)}{p \rho+q(\rho+1)-\rho(\rho+1)(2 \rho+1)} \tag{35}
\end{equation*}
$$

The explicit sextic for $\kappa$ has coefficients of the third order in $p, q$ and only five coefficients vanish. It is not as convenient for computation as the chain above.

Since only the ratios of bisectors are used in this we may choose the scale so that $(b-c)=1$. The isosceles triangles do not enter in the equation so that this is permissible. We then have $a=\frac{1}{\kappa}$.

As in case I we may conclude that $a$ is the root of an irreducible sextic of the symmetric group $G_{720}\left(\mathrm{I}, \S_{15}\right)$.

## 12. TIIE CASE OF EQUAL BISECTORS (EXTERNAL)

The group-theory argument of $I, \S 20$ may be repeated in this case with the same result. The sextic for the side $a$ remains a sextic when $\bar{K}=\bar{L}$ and then gives three $a$ 's and three $b$ 's. The sextic for $c$ reduces to the square of a cubic but the rational relation fails and it is necessary to solve a quadratic also.

The explicit determination of $a$ which involves the solution of a sextic solvable by a chain of a cubic and quadratic is conveniently performed by a special method.

If $k=l=\mathrm{I}$ and $m$ be taken as the single parameter of the problem and we write $a+b=t$ $a b=u$ the fundamental equations (II) give easily

$$
\begin{equation*}
\frac{(a-b)^{2}}{(1-2 a)(1-2 b) a b}=\frac{m(t+a-1)^{2}}{(2 t-1)(1-t)(1-2 a) a}=\frac{m(t+b-1)^{2}}{(2 t-1)(1-t)(\mathrm{I}-2 b) b} \tag{37}
\end{equation*}
$$

Equating the first fraction to the arithmetic mean of the second and third

$$
\begin{equation*}
2\left(t^{2}-4 u\right)(\mathrm{I}-t)(2 t-\mathrm{I})=m\left[-2 t^{4}-4 u^{2}+5 t^{3}+u t-4 t^{2}+t\right] \tag{38}
\end{equation*}
$$

and the equation $D_{2}(a, b)=0$ is

$$
\begin{equation*}
2 t^{3}+4 u t-5 t^{2}-3 u+4 t-\mathrm{I}=0 \tag{39}
\end{equation*}
$$

From these we have the cubic

$$
\begin{equation*}
m(t-1)(2 t-1)^{2}=(4 t-3)\left(4 t^{2}-5 t+2\right) \tag{40}
\end{equation*}
$$

giving $t$, while (39) gives $u$ as a rational function of $t$ and the sides are obtained by the quadratic and linear equations

$$
a+b=t, a b=u, \text { and } c=\mathrm{I}-a-b
$$

Given two bisectors at one vertex and another.
In this case the symmetry being destroyed a special unsymmetric method is used. The cases where the third bisector is internal or external are reached by a change of sign in a side and can be treated in one set of equations.

As data we take

$$
\begin{equation*}
p=\frac{\dot{K}}{\bar{K}}, q=\frac{K}{L} \tag{I}
\end{equation*}
$$

If $A, B, C$, are the angles of the triangle the fundamental formulas give

$$
\begin{equation*}
p=\tan \frac{B-C}{2}, q=\sin B \cos \frac{C-A}{2} \div \sin A \cos \frac{B-C}{2} \tag{2}
\end{equation*}
$$

If the angles are determined the remainder of the problem is an affair of ruler and compass.
The problem of determining the trigonometric functions of the angles involves an irrationality of the tenth degree, the root of an equation whose group is the symmetric group.

We write

$$
\begin{equation*}
B-C=\theta, B=2 M, q \cos \frac{\theta}{2}=H \tag{3}
\end{equation*}
$$

and treating $H, \theta$ as assigned $M$ as required

$$
\begin{equation*}
H \sin (4 M-\theta)-\sin 2 M \cdot \sin (3 M-\theta)=0 \tag{4}
\end{equation*}
$$

For the purpose of determining the group we write

$$
e^{i M}=x, 2 I I i=k, e^{-2 i \theta}=h
$$

and obtain the algebraic equation

$$
\begin{equation*}
k x\left(h x^{8}-1\right)-\left(x^{4}-1\right)\left(h x^{6}-1\right)=0, \text { which is obviously irreducible. } \tag{5}
\end{equation*}
$$

2. THE MONODROMIE GROUP

For $h=0$ six roots of the equation are infinite, the other four being given by

$$
x^{4}-k x-1=0 .
$$

This has double roots for four values of $k, y$.

$$
k^{4}=-\frac{4^{4}}{27}, 3 x^{4}+1=0, k=4 x^{3}
$$

For $k=0$ the roots are $+\mathrm{I},+i,-i,-1$ and may be named from this position in this order (7), (8), (9), (0).

If $k=\rho e \frac{-\pi c^{1}}{4}$ and $\rho$ decrease from o to $\frac{-4}{(27)^{\frac{1}{4}}}$ the roots (7), (8) approach and (9), (0) depart from the origin in the $y$ plane. The path is for $y=r e^{i \theta}$.

$$
r^{4}\left(\mathrm{I}+2 \sin ^{-} 2 \theta\right)=\mathrm{I}
$$

in rectangular co-ordinates

$$
\left(x^{2}+y^{2}\right)\left(x^{2}+4 x y+y^{2}\right)=1 .
$$

At the extreme value mentioned for $\rho,(7)$ and (8) are equal and give rise to a two-cycle. By moving $k$ along the same radius to the corresponding positive value of $\rho,(9)$ and (0) become equal.

By using the perpendicular radius as a path for $k,(0),(8)$ and (9), (7) respectively become equal in the two directions, the path being

$$
r^{4}(1-2 \sin 2 \theta)=1
$$

Hence we get all interchanges on (7), (8), (9), (0).
Returning to the six infinite roots we find that they form a six-cycle for $h=0$ for all finite $k$ 's.
For $k=\infty$ of (7), (8), (9), (0) three become infinite and one approaches zero. This is obviously the negative root of $x^{4}-k x-1=0$ and was named ( 0 ) in the original position.

For the approach $k=\infty$ followed by $h=0$ nine of the roots are infinite and eight form a cycle. Since every interchange of $(7),(8),(9),(0)$ is allowed we take this cycle as

$$
[(1),(2),(3),(4),(5),(6),(7),(8)] \text { in some order. }
$$

Transforming $[(0),(7)]$ by the powers of this substitution we have with $[(0),(9)]$ every $[(0),(n)]$ and so the symmetric group.

## 3. THE EQUATION FOR THE TANGENT OF A HALF-ANGLE

Writing $y=\tan M, t=\tan \theta$, and $k=2 H i$ the equation (4) becomes

$$
\begin{equation*}
H^{2}\left(y^{2}+1\right)\left(t y^{4}+4 y^{3}-6 y^{2} t-4 y+t\right)^{2}-4 y^{2}\left(y^{3}-3 y^{2} t-3 y+t\right)^{2}=0 \tag{6}
\end{equation*}
$$

The coefficients are rational in $(k, h, i)$ and $y$ is rational in $(x, i)$. The group is also the symmetric group.

For every $y$ there are two $x$ 's but ten belong to $F(x,-k, h)=0$. These in the real cases correspond to values of $M$ increased by $\pi$, and lead to the same triangles. So for a change of sign in $p, t$ and $y$ change also and the same triangles occur, and without loss of generality we may take $p, q$ as positive.

By differentiation and elimination of $H^{2}$ we obtain as a discriminantal equation

$$
\begin{equation*}
t y^{9}-6 y^{8} t^{2}-15 t y^{7}+\left(t^{2}-20\right) y^{6}+21 y^{5}-21 t^{2} y^{4}-21 t y^{3}+\left(3 t^{2}-12\right) y^{2}+6 y-1=0 \tag{7}
\end{equation*}
$$

Treated as a quadratic in $t$ the discriminant is

$$
\left(y^{2}+1\right)^{6}\left(y^{4}-36 y^{2}-12\right)
$$

Hence real roots only occur for $|y|<6.02 \ldots$.
The real discriminantal curve is then determined rationally in $y$ and a square root of a function of $y$. It may be traced by assigning to $y$ all real values outside the critical values and calculating $t, I I$ and so $p, q$. The curve has quadrantal symmetry in the $(p, q)$ plane. The critical values correspond to $p=.1456 \ldots q=.2851 \ldots$

From this value $t=.2974$. . one value of $t$ increases monotonously to $\infty$ and the other decreases monotonously to zero. The $p, q$ values tend each monotonously to $(\mathrm{I}, \mathrm{o})$ and $\left(0, \frac{1}{2}\right)$ respectively.
. There is a cusp at $\left(0, \frac{1}{2}\right)$ where $t \propto \frac{\mathrm{I}}{y^{3}}, p \propto \frac{\mathrm{I}}{y^{3}}$, and $q-\frac{1}{2} \propto \frac{\mathrm{I}}{y^{2}}$. At ( $\mathrm{I}, \circ$ ) the proper approximation is a pair of parabolas.


The curve is continued past $p=1$ by negative values of $y$ and $t$, which double change of sign leaves $I^{2}$ unchanged while $(p, q)$ are transformed to $\left(\frac{1}{p}, \frac{q}{p}\right)$, the same transformation as that effected by the interchange of two of the assigned bisectors, or say of $(f, g)$ where

$$
\mathrm{I}: p: q:: \frac{1}{f}: \frac{1}{g}: \frac{1}{h} .
$$

In addition to the real branches thus traced there is a conjugate point $p=\infty, q=0$ (Fig. 23).
4. THE SOLUTION OF THE REAL PROBLEM

The graph of

$$
H=q \cos \frac{\theta}{2}=\sin 2 M \sin (3 M-\theta) \div \sin (4 M-\theta)
$$

is of the same general character for any $\theta$. The diagram is for $\theta=\frac{\pi}{4}$ (Fig. 24).
In general the zeros are : $0, \frac{\pi}{2}, \pi, \frac{3}{2} \pi$ independent of $\theta$, and $\frac{\theta}{3}+\frac{m \pi}{3}, m=1,2, \ldots 6$.
The infinities are $\frac{\theta}{4}+\frac{m \pi}{4}, m=1,2, \ldots 8$.


As $p$ ranges from $\circ$ to $\infty, \theta$ ranges from $\circ$ to $\pi$, and these values coalesce only for $\theta=$ $\frac{\pi}{2}$ when $\frac{\theta}{3}+\frac{\pi}{3}=\frac{\pi}{2}$. This corresponds to $p=1$ and two zeros coalesce but remain real on passing the value, (3) and (4) being interchanged.

For given $H$ the roots are either so real or 8 real. The critical values can be found from the derivative vanishing at the roots of

$$
2 \cot 2 M-4 \cot (4 M-\theta)+3 \cot (3 M-\theta)=0
$$

These values can be found without much trouble from the table of natural cotangents.

The values of the roots which are then entirely separated can be found either by Horner's method from the equation (6) or by trial from the table of logarithmic sines, and

$$
\begin{equation*}
\log H=\log \sin 2 M+\log \sin (3 M-\theta)-\log \sin (4 M-\theta) \tag{8}
\end{equation*}
$$

The triangles are all real and possible if $y$ or $M$ is real, for in all cases $A+B+C=\pi$. The values of $A, B, C$ are not always positive and the results are subject to an interpretation by way of interchanging internal and external bisectors.

## 5. the character of the solutions

First take $o<p<1$, i.e. $0<\theta<\frac{\pi}{2}$.
For the root ( I )

$$
0<M<\frac{\theta}{4}: 0<B<\frac{\theta}{2}:-\theta<C<-\frac{\theta}{2}:-\theta<B+C<0 .
$$

Since $\sin A=\sin (B+C)$ we have $a<0, b>0, c<0$ and the solution refers to $\bar{K}, K, L$ in place of $K, \bar{K}, L$.

For the root (2)

$$
\frac{\theta}{3}<M<\frac{\theta}{4}+\frac{\pi}{4}: \frac{2 \theta}{3}<B<\frac{\theta}{2}+\frac{\pi}{2}:-\frac{\theta}{3}<C<\frac{\pi}{2}-\frac{\theta}{2}: \frac{\theta}{3}<B+C<\pi .
$$

Hence $a>0, b>0$, and $c$ is of doubtful sign. For $q<p, c<0$.
So for $q<p(2)$ gives $a, b, c:+,+,-$ and refers to $\bar{K}, K, \bar{L}$ and for $q>p, a, b, c:+,+$, + and refers to $K, \bar{K} L$ that is to the original verbal statement of the problem.

The roots (3), (4) are alike and have $a, b, c:-,+,+$ and refer to $K, \bar{K}, \bar{L}$.
For the root (5)

$$
\frac{\theta}{4}+\frac{\pi}{2}<M<\frac{\theta}{3}+\frac{2 \pi}{3} .
$$

Here $a$ is always positive, $b$ negative, and $c$ changes sign when $C=\pi$ or when $q=\mathrm{I}$.
The other cases are of invariable class and the results may be collected:
I. The original case $a, b, c:+,+,+$ occurs for (6) and (2) if $q>p$.
II. The case $a, b, c:+,-,-$ occurs for (3), (4), (8), (10) and (5) if $q<\mathrm{I}$.
III. The case $a, b, c:+,-,+\operatorname{occurs}$ for (5) if $q>\mathrm{I}$ and for ( I ).
IV. The case $a, b, c:+,+,-$ occurs for (7), (9).

The cases where $p>I$ can be included by noticing the transformation $\left(p, \frac{1}{p}\right)$ combined with $\left(q, \frac{q}{p}\right)$ which entails $(\theta, \pi-\theta)$ and if also we interchange $(M, \pi-M)$ the original equation is unchanged.

This interchange however takes $\sin B$ to $-\sin B$ and leaves $\sin A, \sin C$ invariant. The triangles are unchanged but the internal and external bisectors at $A$ are interchanged. The classes of solutions (I, III) and (II, IV) are interchanged in the pairing given.

The whole transformation is equivalent to an interchange of the fundamental quantities $K$, $\bar{K}$, or if $K: \bar{K}: L:: f: g: h$ to $(f, g)$.

Under ( $g, h$ ) which involves $(p, q)$, however, a new problem arises, and so under ( $f, h$ ) which replaces $p$ by $\frac{p}{q}$ and $q$ by $\frac{1}{q}$.

Of the six permutations of $(f, g, h)$ three sets of two lead to distinct sets of triangles.
Namely for this problem

$$
\begin{array}{rlrl}
(f, g, h) & \equiv(g, f, h) \text { corresponds to } & (p, q) & \equiv\left(\frac{1}{p}, \frac{q}{p}\right): \\
(h, g, f) \equiv(g, h, f) & \left(\frac{p}{q}, \frac{1}{q}\right) & \equiv\left(\frac{q}{p}, \frac{1}{p}\right) . \\
(h, f, g) \equiv(f, h, g) & (q, p) & \equiv\left(\frac{1}{q}, \frac{p}{q}\right)
\end{array}
$$

## 6. THE CASE OF EQUAL BISECTORS

For $p=q=I$ the equation for $x=\tan \frac{B}{2}$ becomes

$$
\left(x^{2}+1\right)\left(x^{4}-6 x^{2}+1\right)-8 x^{2}\left(3 x^{2}-1\right)^{2}=0
$$

which reduces to

$$
\begin{equation*}
\left(x^{2}-1\right)\left(x^{4}-14 x^{2}+1\right)\left(x^{4}+4 x^{2}+1\right)=0 \tag{9}
\end{equation*}
$$

The first factor gives in two ways the triangle $A=B=\frac{\pi}{2}, C=0$.
The second factor gives in four ways the triangle $A=B=\frac{\pi}{6}, C=\frac{2 \pi}{3}$.
The third factor gives in four ways the triangle determined by

$$
B=2 \tan ^{-1} \sqrt{(\sqrt{5}-2)}
$$

or approximately when the angles are taken positively and internal,

$$
A=13^{\circ} 40^{\prime}, B=\mathbf{1} 28^{\circ} 10^{\prime}, C=38^{\circ} 10^{\prime}
$$

In this case the $B$ bisector is external, in the second case it is internal, and in the first the words internal and external have no proper distinction. The case of $p=1, q$ any, has an equation containing only even powers of $x$. If the problem were solved in terms of the sides the locus $p=I$ would be discriminantal, but for this equation, although the group reduces so that the equation may be solved by solving a quintic and quadratic, there are no equal roots, the roots merely referring to the same five triangles in pairs. In the former cases we had the phenomenon of a discriminantal locus in one solution corresponding to a locus of reducibility for another; here we have it corresponding to a locus of group reduction.

## IV

## I. THE GENERAL PROBLEM FOR REAL DATA

If three real numbers are assigned as the lengths of any three bisectors the problem of determining the triangle is to be solved by successive application of the methods of the three cases. The number of real solutions depends on the data, and the character of the dependence is revealed by considering the discriminants of the three cases simultaneously.

In cases I and II any three assigned real numbers cause the ( $\alpha, \beta$ ) point to fall in the region within the cusp of $D(a, \beta)$. For $a>4$ I has 8 real solutions with possible triangles; II has 4 . For $\alpha<4$ the numbers are 7 and o respectively.

The condition $a=4$ is expressed in $(p, q)$ as the vanishing of the product

$$
(p+q+I)(p+q-I)(p-q+I)(-p+q+I) .
$$

On account of the symmetry of the discriminant of III it is only necessary to consider one quadrant of the ( $p, q$ ) plane (Fig. 24).

Taking the first quadrant for $p+q-1>0$ we have $a<4$. There are then three regions and three classes of the general problem:

> Class $A: a<4$ and $\Delta_{3}<0$.
> Class $B: a>4$ and $\Delta_{3}<0$
> Class $C: a>4$ and $\Delta_{3}>0$

For class $A$, I has 7 , II has 0 , and III has 3 permutations, each of which has 8 solutions. The permutations of ( $f, g, h$ ) leave the square ( $\alpha=4$ ) invariant.

The total for class $A$ is then 33 real solutions with proper triangles. For class $B$ the permutations of ( $f, g, h$ ) do not carry the representative point across the discriminantal curve. III has then three sets of 8 solutions, I has 8 , and II has 4 , the total being 36. For class $C q<\frac{1}{2}$ and its reciprocal occurring as a $q$ under $(h, f)$ is outside $\Delta_{3}$. The other transform is inside $\Delta_{3}$.

For this class two sets in III have io real solutions and I has 8, and II 4 solutions : in all, 40. This is the greatest number and occurs, for example, if $f: g: h:: 3: 30: 10$ (Fig. 23).

This case has been taken for the triangles in the illustration (Figs. 25, 26, 27).

## 2. THE PROBLEM WHEN A RIGIIT ANGLE AND TWO BISECTORS ARE GIVEN

Taking the right angle as $C$ the sides $b$ and $c$ are rational functions of the side $a$.

$$
\begin{equation*}
b=\frac{2 a-1}{2(a-1)}, \quad c=\frac{-2 a^{2}+2 a-1}{2(a-1)}, \quad a+b+c=1 \tag{I}
\end{equation*}
$$

By interchange of $(a, b)$ and by changing the signs of sides the fifteen pairs of the six bisectors can be reduced to three cases.


## Case I. Given $K, L$.

The ratio $\frac{L^{2}}{K^{2}}$ becomes a perfect square in $\left(a, V^{2}\right)$ namely

$$
\begin{equation*}
\frac{L}{K}=2 V / 2 a(a-1)^{2} \div(1-2 a)^{2} \tag{2}
\end{equation*}
$$

If $L \div 2 \sqrt{2} K$ is plotted against $a$ the same curve which occurred in the internal problem (I, $\$ 8{ }^{7}$ is given.

For $\frac{L}{K}>0$ there is then one real solution of the cubic. This gives a real triangle with all positive sides and is the solution of the problem as stated, namely $K, L$ are internal bisectors.

For $\frac{L}{K}<0$ there are three solutions:

1) $a>\mathrm{I}, b>_{\mathrm{I}}, c<0$. The range of $c$ has a maximum at $c=-1 / 2-\mathrm{I}$ corresponding to the right-angled isosceles triangle and $c$ tends to $-\infty$ as either $a$ or $b$ tends to $\infty$.

The bisectors are both external.
2) $\mathrm{I}>a>\frac{1}{2},-\infty>b>0, \infty>c>\frac{1}{2}$. The $A$ bisector is external, the $B$ bisector internal.
3) $0>a>-\infty, \frac{1}{2}<b<\mathrm{I}, \frac{1}{2}<c<\infty$. The $A$ bisector internal, the $B$ bisector external.

In all there is a single solution for each arrangement of the bisectors as internal or external. The discriminant of the cubic is

$$
\begin{equation*}
\frac{L\left(8 \sqrt{ } / 2 L^{2}+13 L K+8 V^{/} 2 K^{2}\right)}{8 K^{3}} \tag{3}
\end{equation*}
$$

The double points are at $L=0, K=0$ and $\frac{L}{K}=\frac{[-13 \pm V /(-7)]}{16 \rho^{-2}}$.
Case II. Given $K, M$.
The equation for the side $a$ is a sextic; obviously irreducible.

$$
\begin{equation*}
8 K^{2}(a-1)^{4} a^{2}-M^{2}\left(2 a^{2}-1\right)^{2}\left(2 a^{2}-2 a+1\right)=0 \tag{4}
\end{equation*}
$$

Plotting $\frac{8 K^{2}}{M^{2}} \equiv p$ against $a$ we have the real graph (Fig. 28). At $p=0$ a double transposition can be effected ( $\mathrm{I}, 2$ ) ( 3,4 ) and at $p=\infty$ a two-cycle and a four-cycle which must separate 5,6 which are conjugate complex for the real approach. It may properly be denoted by $(2,3)(1,5,4,6)$. Approaching $p=\infty$ from the negative side no double point is encountered between $p=0$ and $p=-\infty$ since the double points are at $p=0, \infty$ and the four complex roots of the remaining factor of the discriminant:

$$
16 p^{4}-152 p^{3}+93 p^{2}+512 p+32768
$$

The corresponding values of $a$ being given by

$$
6 a^{4}-8 a^{3}+7 a^{2}-4 a+1=0 .
$$

Since the conjugate pairing must be kept the cycle at $p=\infty$ must be for this approach either $(\mathrm{I}, 2)(3,5,4,6)$ or $(3,4)(\mathrm{I}, 5,2,6)$ or $(5,6)(\mathrm{I}, 3,2,4)$.

Since the complex double points are distinct and the second derivative does not vanish at them, a single transposition occurs as an element of the monodromic group. It is then easy to see that the monodromie group and therefore the algebraic group is the symmetric group,

however the cycle for $p=-\infty$ be named. The discriminant not being a square, adjunction of $1 / \bar{p}$ does not reduce the group.

To discuss the character of the solutions we divide into classes by $p \gtrless 8$ that is $K \gtrless M$ and by the values for $a$ when $p=8$.

Class I $a . \quad K<M, a<-\frac{\mathrm{I}}{1 \frac{1}{2}}, \mathrm{r}>b>\frac{\mathrm{I}}{1 \frac{-}{2}}, c<\mathrm{I} . \quad M$ is external.
Class II $a . K<M,-.2516 \ldots>a>-\frac{\mathrm{I}}{\mathrm{l}^{-\frac{1}{2}}}, .6005 \ldots<b<\frac{\mathrm{I}}{\mathrm{I}^{-\frac{2}{2}}}, .65 \mathrm{II},<\mathrm{c}<\mathrm{I} . \quad M$ is external and the general character of the figure is the same as in $I a$.

Class III $a . K<M, \frac{\mathrm{I}}{\sqrt{2}}>a>\frac{1}{2},-\frac{\mathrm{I}}{1^{-2}}<b<0, \mathrm{I}>c>\frac{1}{2}$. $K$ and $M$ are both external.
Class IV $a$. $K<M, \frac{1}{\sqrt{\frac{2}{2}}}<a<.7600 \ldots,-\frac{\mathrm{I}}{1 \frac{1}{2}}>b>-$ ri. $083 \ldots, \mathrm{r}<c<\mathrm{I} .323 \ldots, K$ and $M$ both external.

Class Ib. $K>M,-.2516<a<0, .6005>b>\frac{1}{2}, .651 \mathrm{I}>c>\frac{1}{2} . \quad M$ is external, $K$ internal.
Class IIb. $K>M, 0<a<\frac{1}{2}, \frac{1}{2}>b>0, c$ has the value $\frac{1}{2}$ at each end of the range and
 isosceles triangle.
$K$ and $M$ are both internal and this case is the only one solving the problem verbally expressed for internal bisectors.

Class III $b$. $K>M, .7600 \ldots<a<\mathrm{I},-\mathrm{r} .083_{3}>b>-\infty, \mathrm{I} .323 \ldots<c<\infty$. Both bisectors are external.

Class IV $b$. $K>M, 1<a<\infty, \infty>b>\mathrm{I}, c$ has the value $-\infty$ at each end of the range and reaches a maximum $-1 / 2-I$ for the case of the isosceles triangle. $K$ is external.

The approximate values for $a$ are the roots of the equation for equal bisectors. For this case only two real non-trivial solutions exist:
$a: b: c::-.2516$. . : . 6005 . . : . 6511 . . The angle $A$ about $22^{\circ} 40^{\prime}, K$ internal, $M$ external.
$a: b: c:: .7600 \ldots-1.083$, ,: 1.323.., $A$ about $35^{\circ} 4^{\prime}, K$ and $M$ both external. Case III. Given two bisectors at one vertex and the right angle.

The tangent of half the difference of two angles, and one angle are given, hence the problem is one for ruler and compass.

## 3. SPECIAL CASES OF ISOSCELES TRIANGLES

The general method of Case II leaves the construction of an isosceles triangle given an external bisector at the base indeterminate.

Other conditions must be given.
If the base $a$ and the external bisector $\bar{L}$ are given we have

$$
\bar{L}^{2}=\frac{(-a+2 b) a^{2} b}{(b-a)^{2}} .
$$

To determine the angles write $\frac{a}{\bar{L}}=p=\frac{\sin 3 \phi}{\sin 2 \phi}$ where $\phi$ is half the external angle at the base.




III. 66

The solution is

$$
\cos \phi=\frac{p \pm 1 \overline{\left(p^{2}+4\right)}}{4}
$$

The problem cau be solved by ruler and compass and is an extension of Euclid's decagon problem ( $p=-1$ ).

The sign of $p$ does not determine any representable difference of configuration, but for $\frac{1}{\sqrt{2}}<|p|<\frac{3}{2}$ one triangle has the bisector internal : below the lower value two solutions have external bisectors, above the higher one triangle is complex.

If the sides $b=c$ and $\widetilde{L}$ be given the problem requires the solution of a cubic equation. Namely if $\frac{\bar{L}^{2}}{b^{2}}=\kappa$, and $\frac{a}{b}=\lambda$,

$$
\lambda^{3}+(\kappa-2) \lambda^{2}-2 \kappa \lambda+\kappa=0 .
$$



The discriminant is $\kappa\left(4^{\kappa^{2}}+13^{\kappa}+3^{2}\right)$, and $\kappa=\infty$ is also discriminantal.
For $\kappa>0$ there are three real roots, one negative referring to an internal bisector and two positive referring to external bisectors. For $\kappa=I$ the angle $A$ is $\frac{3 \pi}{7}$ and the bisector internal, or $A$ is $\frac{5 \pi}{7}$ or $\frac{\pi}{7}$ and the bisector external.
4. the identical relations among the six bisectors
I. We have $\frac{K}{\bar{K}}=\tan \left(\frac{B-C}{2}\right)$,
whence

$$
\begin{equation*}
\Sigma\left(\frac{K}{K}\right)=\Pi\left(\frac{K}{K}\right) \tag{5}
\end{equation*}
$$

II. The length of the line joining the extremities of the two bisectors from $A$ is $\frac{2 a b c}{b^{2}-c^{2}}$. Hence

$$
\begin{equation*}
\sum_{1} \frac{1}{\left(K^{2}+K^{2}\right)}=0 \tag{6}
\end{equation*}
$$

III. The altitude of the right triangle included by the bisectors at $A$ and the line joining their extremities is

$$
H_{a}=\frac{K \cdot \bar{K}}{{ }_{2} 1 \frac{\left.K^{2}+\bar{K}^{2}\right)}{(a n d ~ a s ~ a l t i t u d e ~ o f ~ t h e ~ t r i a n g l e ~} H_{a}=\frac{S}{2 a}, \text {, }} \text { and }
$$

where $S$ is the area of the triangle.
From this equality

$$
\begin{equation*}
a=S \cdot \frac{\sqrt{\left(K^{2}+\bar{K}^{2}\right)}}{K \cdot \bar{K}} \tag{7}
\end{equation*}
$$

while

$$
\frac{K^{2}}{\overline{K^{2}}}=\frac{s \cdot(s-a)(b-c)^{2}}{(s-b)(s-c)(b+c)^{2}}
$$

By substituting for $a, b, c$ from $(7)$ in $\frac{\Sigma^{K^{2}}}{K^{2}}$ a third relation is obtained.
This is for convenience expressed by writing

$$
K^{2}=\frac{\mathrm{I}}{\kappa}, \bar{K}^{2}=\frac{\mathrm{I}}{\kappa^{\prime}}, \text { etc. }
$$

and
when it takes the form

$$
p^{2}=\kappa+\kappa^{\prime}, q^{2}=\lambda+\lambda^{\prime}, \text { etc. }
$$

$$
\begin{equation*}
\Pi_{\frac{\kappa^{\prime}}{\kappa}}^{\kappa}=\frac{(\Sigma p)^{3} \Pi(p-q)^{2}}{\Pi(p+q-r) \Pi(p+q)^{2}} \tag{8}
\end{equation*}
$$

It is to be noticed that $p: q: r:: a: b: c$.
The independence of the conditions I and II is obvious. For III the set of values

$$
K=2, \overline{5}=\bar{K}, L=41^{2}, \bar{L}=21^{\prime}, M=3, \bar{M}=1
$$

satisfy I and II but not III.

## 5. THE INDIRECT PROOF

In a series of papers in Phil. Mag., IV (1852), the problem of the triangle with two equal angle-bisectors is made the text (with some other elementary problems) of a discussion as to the necessity of the reductio ad absurdum in geometry. The chief parts were taken by Sylvester and De Morgan.

De Morgan claims to see "identity in 'Every $A$ is $B$ and every not $B$ is not $A$ "'" by a process of thought prior to syllogism; and so denies the necessity of an indirect proof in any case.

Sylvester surmised that "The reductio ad absurdum not only is of necessity to be employed, but moreover in propositions of an affirmative character, need never be employed except when the analytic demonstration is founded on the impossibility or inadmissibility of certain roots due to the degree of the equation implied in the conditions of the question. If this surmise turn out to be correct we are furnished with a universal criterion for determining when the use of the indirect method of geometrical proof should be considered valid and admissible and when not."

It is difficult to deny De Morgan's general proposition though his immediate application is a little unfortunate. The problem being, as stated by Sylvester, "To prove that if from the
middle of a circular arc two chords be drawn, and the remoter segments of these chords cut off by the line joining the end of the arc be equal, the nearer segments are equal." The doubtful word is of course "remoter." If this word means every point of which is remoter, then De Morgan's contention that "proving that the inequality of the nearer segments makes the inequality of the remoter ones follow, the equality of the remoter ones makes the equality of the nearer ones follow" is a proper special case of his general argument, can be made good. This interpretation has however a disadvantage from the geometric point of view. It is not applicable to the allied problem where the analytic geometer would say the chord has complex points of intersection with the circle. Yet for this problem an entirely analogous theorem is true and it is desirable to so state the problem that both cases are included.

If the problem be stated: "A line of given length has one extremity on a straight line, the other on a circle, and the line passes through a cut of the circle and the perpendicular on the given line from the center of the circle which is not separated from the foot by the second cut; then if the length of the line be less than the distance from foot to second cut, and in case the foot is outside the circle greater than the mean proportional between twice the distance from

- foot to second cut and the distance from foot to first cut, and greater than the mean proportional between four times the diameter of the circle and the distance from foot to first cut, four positions are possible for the line, and these have symmetry in pairs, and for each symmetrical pair the segments cut from the line by the circle are equal"-then it is possible that justice has been done to the facts. However, in the general case the segments by the circle are neither nearer nor remoter and from the inequality of the circle segments the inequality of the circle line segments does not follow without a specification of the pairing. The syllogist's difficulty lies in the definition of the classes, and in this special case the class is at least not conveniently defined by equations alone.

Turning to Sylvester's view we note that the proof that cqual internal bisectors implies isoscelism falls very neatly in his scheme but the corresponding problem for external bisectors presents a new difficulty. Sylvester with the proper mathematical instinct generalized the problem before solving it: ${ }^{\text { }}$ namely, he said divide the internal angle in a given ratio instead of bisect. This generalization unfortunately does not include external bisection. To compare the two cases we write the equation

$$
\frac{K-L}{b-a}=\frac{c}{K+L} \cdot \frac{a+b+c}{b+c}\left[\frac{c}{a+c}+\frac{b(a+b+2 c)}{(a+c)^{2}} \times \frac{a}{b+c}\right] .
$$

This holds from the fundamental equations for internal bisectors, and the spirit of Sylvester's method is to say-

The right-hand side is essentially positive for a-positive non-trivial triangle, and is moreover expressed in products and sums of products of ratios each geometrically interpretable. If then $K-L=0, b-a=0$ or the axiom of Archimedes fails.

In the external case, however,

$$
\frac{\bar{K}-\bar{L}}{b-a}=\frac{a+b-c}{\bar{K}+\bar{L}} \frac{c}{(b-c)}\left[\frac{c^{3}-(a+b) c^{2}+3 a b c-a b(b+a)}{(b-c)(c-a)^{2}}\right]
$$

the last factor in the numerator may vanish for positive non-trivial triangles. It is in fact
${ }^{1}$ Blichfeldt, Annals of Maths., II, 4.22, gives the same gencralization. His proof is valid also for nonEuclidean space.
$-D_{2}(a, b)$ and the curve $D_{2}=0$ actually enters the region of proper triangles in the $(a, b, c)$ plane (Fig. I8).

In this case, however, the non-isosceles triangles with equal external bisectors have the bisectors oppositely directed so that further specification of the conditions of the problem which must again presumably be by means of inequalities and not equations will permit the proof as above by Archimedes' axiom.

It would appear that in general, though the difficulties of expression may be great, any theorem true analytically for a properly restricted class might conceivably be thrown into a form similar to the above and further a direct geometric proof might be given by Archimedes' axiom and adequate restrictions based on order postulates.

## 6. GENERALIZATIONS OF THE PROBLEM

Sylvester's generalization (loc. cil.) which substitutes division of the angle in a given ratio for bisection does not include the external case as well as the internal under the same general formulas. The same thing is true if for bisectors which meet sides in points dividing them in the ratio of adjacent sides we substitute lines through the vertices dividing opposite sides in a ratio compounded of the ratio of adjacent sides and the ratio of a corresponding pair of three assigned numbers.

To Professor E. H. Moore is due a generalization embracing both the internal and external cases in one set of formulas.

He introduces three parameters $u, v, w$ and defines the given quantities by

$$
\begin{aligned}
& K_{a}^{2}(u, v, w ; a, b, c) \equiv \frac{(a u+b v+c w)(-a u+b v+c w) b c v w}{(b v+c w)^{2}} \\
& K_{b}^{2}(u, v, w ; a, b, c) \equiv K_{a}^{2}(u, v, w ; b, c, a) \\
& K_{c}^{2}(u, v, w ; a, b, c) \equiv K_{c}^{2}(u, v, w ; c, a, b)
\end{aligned}
$$

Then for $(u, v, w)=(\mathbf{1}, \mathrm{I}, \mathrm{I})$ the internal formulas are given and for $(u, v, w)=(\mathrm{I}, \mathrm{I},-\mathrm{I})$ the external formulas.

The problem for the spherical triangle, the formulas are

$$
\tan ^{2} K=\frac{4 \sin s \sin (s-a) \sin b \sin c}{\sin ^{2}(b+c)}, \quad \text { etc. }
$$

It may be noted that the dual spherical problem reduces as the sphere becomes a plane to a ruler and compass problem. Given the angles which the medians make with the sides to construct the triangle.


## VITA

Richard Philip Baker was born February 3, 1866, at Condover, Shropshire, England, and was educated at Clifton College, Bristol (1877-84), and at Balliol College, Oxford (1884-87). He graduated with the degree of B.Sc. at the University of London in 1887.

In 1888 leaving England for the United States he studied law and was admitted to the Texas bar in 1890. In 1895 he became a graduate student of mathematics at the University of Chicago where he attended courses by Professors Young, Laves, Maschke, Bolza, Moore, Dickson, and Wilczynski. After several years of teaching in secondary schools he became in 1905 instructor in mathematics at the State University of Iowa. Here he studied physics with Professors Guthe and Stewart. In 1910 he was appointed assistant professor of mathematics in the same university.


[^0]:    ${ }^{1}$ See Barbarin in Mathesis (1896), 143-60; also Bull. de. S.M.F. (1894), 76-80.
    ${ }^{2}$ Pet. Mem., XI (1765).
    ${ }^{3}$ L'Intermédiaire (1894), 1-149; also in Zeitschrifl f. M. u. N. U., 32.444.

